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Non-renormalizability of θ-expanded non-commutative QED

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Abstract: Computing all divergent one-loop Green's functions of θ-expanded non-commutative quantum electrodynamics up to first order in θ, we show that this model is not renormalizable. The reason is a divergence in the electron four-point function which cannot be removed by field redefinitions. Ignoring this problem, we find however clear hints for new symmetries in massless θ-expanded non-commutative QED: four additional divergences which would be compatible with gauge and Lorentz symmetries and which are not reachable by field redefinitions are absent.

Keywords: Renormalization Regularization and Renormalons, Gauge Symmetry, Non-Commutative Geometry.

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Contents

1. Introduction

The investigation of field theories on a very special type of non-commutative spaces — the so-called non-commutative $\mathbb{R}^4$ involving a constant antisymmetric tensor $\theta^{\mu\nu}$ — became during the last two years an industry, making it impossible to list all relevant contributions. Fortunately one can refer to some reviews [1]–[4], which reflect most of the achievements. Nevertheless, let us briefly recall a few selected milestones which lead directly to the question addressed in this paper.

Many investigations on this subject are motivated by the discovery of compactifications of M theory on non-commutative tori [5] and the identification of these non-commutative geometries as limiting cases of string theory [1]. Quantum field theories on θ-deformed space-time were shown to be ultraviolet divergent [6] (motivated at this time by the continuum version of the twisted Eguchi-Kawai model [7] which already possesses non-commutative features). A more general result on divergences for spaces which are non-commutative manifolds in the sense of [8] was obtained in [9]. The crucial question about the physical relevance of field theories on non-commutative space-time is whether these models are renormalizable. At the one-loop level, renormalizability of U(1)-gauge theories on θ-deformed spaces was first proved in [10, 11, 12]. One-loop renormalizability of non-commutative U(n)-gauge theories was proved in [13].[1] At higher loop order there was discovered a new type of infrared divergences both in scalar field theories [15] and Yang-Mills theory [16, 17], which prevents a perturbative renormalization of these models. A power-counting theorem for non-commutative field theories which classifies the divergences was established in [18].

A very surprising result due to Seiberg and Witten [1] was that non-commutative and commutative gauge theories are related by a formal power series in θ, the so-called Seiberg-Witten map. The Seiberg-Witten map combines the θ-dependence through the $\star$-product with a cleverly chosen θ-dependence of the non-commutative gauge fields to produce an action which is (commutatively) gauge-invariant[2] at *any* order n in θ. In this way the Seiberg-Witten map gives rise to a quantum field theory of N-point gauge field Green's functions with up to n factors of θ which is free of any infrared divergences [19].[3] However, since θ has to be regarded as an external field of power-counting dimension -2, the renormalizability of such a theory (up to given order in θ but any order in $\hbar$) is questionable. The fact that the superficial divergences in the photon selfenergy are (to all orders in θ and $\hbar$) field redefinitions [21] gave some hope that gauge theories on θ-deformed space-time could finally be renormalizable via the Seiberg-Witten map. However, it became clear that one cannot proceed along this line without the identification of new symmetries of the θ-expanded action which could restrict the structure of otherwise possible divergent counterterms. The first candidates of such symmetries — Lorentz rotation and dilatation — were shown to give no further information beyond conformal symmetry of a Yang-Mills theory on commutative space-time [22]. As a by-product, however, we have obtained in [22] a deeper understanding of the Seiberg-Witten map: one has to distinguish between "observer Lorentz transformations" (which transform $\theta^{\mu\nu}$ as a tensor) and "particle Lorentz transformations" (which leave $\theta^{\mu\nu}$ invariant), see [23]. Demanding (A) observer Lorentz invariance and (B) gauge invariance of the particle Lorentz symmetry breaking of the physical action, the particle Lorentz transformation of a field is the sum of its naive observer Lorentz transformation and an additional part given by the Seiberg-Witten differential equation (which is very conveniently derived in this manner).

[1] Partial results were obtained in [14].

[2] Expanding only the $\star$-product in θ, gauge invariance of the truncated action is lost.

[3] That work was inspired by [20].

A brute-force approach to probe the existence of additional symmetries is to compute one-loop Green's functions other than the selfenergy. For θ-deformed Maxwell theory (which has no θ-independent interactions) the computational effort becomes tremendous. In this paper we therefore focus on θ-deformed non-commutative QED [24], for which we are able to compute all divergent one-loop Green's functions up to first order in θ. The (already extremely lengthy) computation in analytic regularization [25] is performed using a *Mathematica*^{TM} package [26]. The final result is simple and (at least for some people) disappointing:

$$\textit{Non-commutative QED cannot be renormalized by means of Seiberg-Witten expansion}\,. \tag{1.1}$$

We provide some ideas how the Seiberg-Witten expansion can be used as a computational technique to treat one-loop divergences of the full (θ-unexpanded) non-commutative QED [16], but this leads — similarly as UV/IR mixing — to problems at the two-loop level. This is the end of the chapter on non-commutative quantum field theories treated by the Seiberg-Witten map.

2. Non-commutative $\mathbb{R}^4$

Our presentation of the non-commutative $\mathbb{R}^4$ is (with different notations, however) based on [27] and the appendix of [28]. Let $\mathcal{A}$ be the space of Schwartz functions[4] $a \equiv a(x)$ on $\mathbb{R}^4$, equipped with the multiplication rule[5]

$$(a\star b)(x) := \int_{\mathbb{R}^4}\frac{d^4k}{(2\pi^4)}\int_{\mathbb{R}^4} d^4y\; a\left(x{+}\frac{1}{2}\theta{\cdot}k\right)\, b(x{+}y)\,\mathrm{e}^{\mathrm{i}k\cdot y}\,, \tag{2.1}$$

where $\theta \in M_4\mathbb{R}$ is a real-valued antisymmetric constant matrix, $\theta^{\mu\nu} = -\theta^{\nu\mu}$, and $(\theta{\cdot}k)^\mu = \theta^{\mu\nu}k_\nu$, $k{\cdot}y = k_\mu y^\mu$. Position space variables are denoted by x, y, z and momentum space variables by k, l, p, q, r, s, t. The multiplication (2.1) is associative but non commutative. The $\star$-(anti)commutators are defined by $[a,b]_\star = a\star b - b\star a$ and $\{a,b\}_\star = a\star b + b\star a$. There is an involution on $\mathcal{A}$ given by complex conjugation $a^*(x) = \overline{a(x)}$ which satisfies $(a\star b)^* = b^*\star a^*$. Partial derivatives $\partial_\mu = \partial/\partial x^\mu$ are derivations with respect to (2.1), $\partial_\mu(a\star b) = \partial_\mu a\star b + a\star\partial_\mu b$. Since Schwartz class functions are trace-class one can define an integral

$$\int a = \int_{\mathbb{R}^4} d^4x\, a(x)\,, \qquad \int a\star b = \int b\star a\,. \tag{2.2}$$

An important concept is that of the multiplier algebra

$$\mathcal{M} = \big\{ f : \mathbb{R}^4 \to \mathbb{C},\ \ f\star a \in \mathcal{A}\ \text{ and } a\star f\in\mathcal{A}\ \text{ for all } a \in \mathcal{A}\big\}\,, \tag{2.3}$$

[4] The Schwartz space $\mathcal{S}(\mathbb{R}^4)$ is the space of smooth complex-valued functions a on $\mathbb{R}^4$ such that for all multi-indices α, β there exist constants $C^{\alpha,\beta}$ with $|x^\alpha\partial^\beta a(x)| \leq C^{\alpha,\beta}$.

[5] It would be wrong in general to replace (2.1) by $(a\star b)(x) = (\exp(\frac{\mathrm{i}}{2}\theta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial z^\nu})a(y)b(z))_{x=y=z}$, which for instance yields zero if $a(x)$ and $b(x)$ have disjoint support.

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where the product is given by (2.1). The product of $f,g \in \mathcal{M}$ is defined by associativity of (2.1), $(f\star g)\star a = f\star(g\star a) \in \mathcal{A}$ for all $a\in\mathcal{A}$. The multiplier algebra $\mathcal{M}$ contains, for example, the unit $1\in\mathcal{M}$ and the coordinate functions $x^\mu(y)\equiv y^\mu$, which both do not belong to $\mathcal{A}$. The famous formula $[x^\mu,x^\nu]_\star = \mathrm{i}\theta^{\mu\nu}$ is thus an identity in $\mathcal{M}$ and not in $\mathcal{A}$. The multiplier algebra is the biggest compactification of $\mathcal{A}$.

Additionally we introduce the space $\mathcal{H}=\mathbb{C}^4\otimes L^2(\mathbb{R}^4)$ of square-integrable bispinors $\eta=\{\eta_s(x)\}_{s=1}^4$, equipped with the sesquilinear inner product[6]

$$\langle\xi,\eta\rangle_{\mathcal{H}} = \int d^4x \sum_{s,s'=1}^{4} \overline{\xi_s(x)}(\gamma^0)^{ss'}\eta_{s'}(x)\,, \tag{2.4}$$

where $\left(\gamma^0\right)^{ss'}$ are the matrix entries of the γ^0-matrix. The multiplication (2.1) extends to an involutive action $\mathcal{A}\times\mathcal{H}\to\mathcal{H}$ obtained by componentwise $\star$-multiplication,

$$\begin{aligned}(a\star\eta)_s(x) &:= \int_{\mathbb{R}^4}\frac{d^4k}{(2\pi^4)}\int_{\mathbb{R}^4} d^4y\; a\left(x+\frac{1}{2}\theta\cdot k\right)\,\eta_s(x+y)\,\mathrm{e}^{\mathrm{i}k\cdot y}\,,\\ \langle a\star\xi,\eta\rangle_{\mathcal{H}} &= \langle\xi,a^*\star\eta\rangle_{\mathcal{H}}\,.\end{aligned} \tag{2.5}$$

We regard bosonic fields as distinguished elements $\phi_i\in\mathcal{A}$ to which one assigns a power-counting dimension $\dim(\phi_i)\in\mathbb{Z}$ and fermionic fields are distinguished Grassmann-valued elements $\psi_i\in\mathcal{H}$ with power-counting dimension $\dim(\psi_i)\in\mathbb{Z}/2$. For $\Gamma[\phi_i]$ being a (sufficiently regular) functional of the fields ϕ_i (fermion fields now included) we define the functional derivative

$$\left\langle\tilde\phi_j[\phi_k],\frac{\delta}{\delta\phi_j}\right\rangle\Gamma[\phi_i] \equiv \left\langle\tilde\phi_j[\phi_k],\frac{\delta\Gamma[\phi_i]}{\delta\phi_j}\right\rangle := \lim_{\epsilon\to0}\frac{1}{\epsilon}\left(\Gamma\left[\phi_i+\epsilon\tilde\phi_i[\phi_k]\right]-\Gamma[\phi_i]\right), \tag{2.6}$$

which replaces a field ϕ_j by the fuctional $\tilde\phi_j[\phi_k]$ in a derivational manner. Summation over j in (2.6) is self-understood. We use (2.6) to define functional derivatives with respect to $\theta^{\rho\sigma}$ as well. Allowing for an explicit θ-dependence of the fields U and V we obtain from (2.1)

$$\left\langle\Theta^{\rho\sigma},\frac{\delta(U\star V)}{\delta\theta^{\rho\sigma}}\right\rangle = \left\langle\Theta^{\rho\sigma},\frac{\delta U}{\delta\theta^{\rho\sigma}}\right\rangle\star V + U\star\left\langle\Theta^{\rho\sigma},\frac{\delta V}{\delta\theta^{\rho\sigma}}\right\rangle + \frac{\mathrm{i}}{2}\Theta^{\rho\sigma}\,(\partial_\rho U)\star(\partial_\sigma V)\,, \tag{2.7}$$

where $\Theta^{\rho\sigma}$ has to be constant.

3. Non-commutative quantum electrodynamics

We are going to study θ-expanded non-commutative QED defined by the classical action (on non-commutative level, i.e. before θ-expansion)

$$\begin{aligned}\hat\Sigma &= \left\langle\hat\psi,\left(\mathrm{i}\gamma^\mu\hat D_\mu - m\right)\hat\psi\right\rangle_{\mathcal{H}} - \frac{1}{4g^2}\int \hat F_{\mu\nu}\star\hat F^{\mu\nu}\,,\\ \hat F_{\mu\nu} &= \partial_\mu\hat A_\nu-\partial_\nu\hat A_\mu - \mathrm{i}[\hat A_\mu,\hat A_\nu]_\star\,,\qquad \hat D_\mu\hat\psi = \partial_\mu\hat\psi - \mathrm{i}\hat A_\mu\star\hat\psi\,,\end{aligned} \tag{3.1}$$

[6]We work in Minkowski space with metric $g^{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$. The inner product (2.4) is therefore not positive definite and $\mathcal{H}$ is not a Hilbert space. The γ-matrices fulfill $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu = 2g^{\mu\nu}1_{4\times4}$ and $\gamma^0(\gamma^\mu)^\dagger\gamma^0=\gamma^\mu$.

for the non-commutative gauge fields (photons) $\hat{A}_\mu = (\hat{A}_\mu)^* \in \mathcal{A}$ and the non-commutative fermion field (electrons) $\hat{\psi} \in \mathcal{H}$ to which we assign the power-counting dimensions $\dim(\hat{A}_\mu) = 1$ and $\dim(\hat{\psi}) = 3/2$. Additionally we define $\dim(\theta^{\rho\sigma}) = -2$. The action (3.1) is invariant under infinitesimal gauge transformations $W^G_{\hat{\lambda}}$, observer Lorentz rotations $W^R_{\alpha\beta}$ and translations W^T_τ, see [22, 29]:

$$W^G_{\hat{\lambda}} = \left\langle \left(\partial_\mu \hat{\lambda} - \mathrm{i}[\hat{A}_\mu, \hat{\lambda}]_\star\right), \frac{\delta}{\delta \hat{A}_\mu} \right\rangle + \left\langle \mathrm{i}\hat{\lambda} \star \hat{\psi}, \frac{\delta}{\delta \hat{\psi}} \right\rangle, \qquad \hat{\lambda} = \hat{\lambda}^* \in \mathcal{A}, \tag{3.2}$$

$$W^R_{\alpha\beta} = W^R_{\hat{A}+\hat{\psi};\alpha\beta} + W^R_{\theta;\alpha\beta}, \tag{3.3}$$

$$\begin{aligned} W^R_{\hat{A}+\hat{\psi};\alpha\beta} = &\left\langle \left(\frac{1}{2}\{x_\alpha, \partial_\beta \hat{A}_\mu\}_\star - \frac{1}{2}\{x_\beta, \partial_\alpha \hat{A}_\mu\}_\star + g_{\alpha\mu}\hat{A}_\beta - g_{\beta\mu}\hat{A}_\alpha\right), \frac{\delta}{\delta \hat{A}_\mu} \right\rangle + \\ &+ \Big\langle \Big(x_\alpha \star \partial_\beta \hat{\psi} - \mathrm{i}\theta_\alpha^{\ \rho}\partial_\beta\partial_\rho \hat{\psi} - x_\beta \star \partial_\alpha \hat{\psi} + \mathrm{i}\theta_\beta^{\ \rho}\partial_\alpha\partial_\rho\hat{\psi} + \\ &\qquad + \frac{1}{4}[\gamma_\alpha, \gamma_\beta]\hat{\psi}\Big), \frac{\delta}{\delta\hat{\psi}} \Big\rangle, \end{aligned} \tag{3.3a}$$

$$W^R_{\theta;\alpha\beta} = \left\langle (\delta^\rho_\alpha \theta_\beta^{\ \sigma} + \delta^\sigma_\alpha \theta^\rho_{\ \beta} - \delta^\rho_\beta \theta_\alpha^{\ \sigma} - \delta^\sigma_\beta \theta^\rho_{\ \alpha}), \frac{\delta}{\delta \theta^{\rho\sigma}} \right\rangle, \tag{3.3b}$$

$$W^T_\tau = \left\langle \partial_\tau \hat{A}_\mu, \frac{\delta}{\delta \hat{A}_\mu} \right\rangle + \left\langle \partial_\tau \hat{\psi}, \frac{\delta}{\delta \hat{\psi}} \right\rangle. \tag{3.4}$$

These Ward identity operators satisfy the following commutation relations:

$$\begin{aligned} [W^G_{\hat{\lambda}_1}, W^G_{\hat{\lambda}_2}] &= W^G_{\hat{\lambda}_{12}}, \\ [W^G_{\hat{\lambda}}, W^R_{\alpha\beta}] &= W^G_{\hat{\lambda}_{\alpha\beta}}, \qquad [W^G_{\hat{\lambda}}, W^T_{\alpha\beta}] = W^G_{\hat{\lambda}_\tau}, \\ [W^R_{\alpha\beta}, W^R_{\gamma\delta}] &= g_{\alpha\gamma}W^R_{\beta\delta} - g_{\beta\gamma}W^R_{\alpha\delta} - g_{\alpha\delta}W^R_{\beta\gamma} + g_{\beta\delta}W^R_{\alpha\gamma}, \\ [W^R_{\alpha\beta}, W^T_\tau] &= g_{\alpha\tau}W^T_\beta - g_{\beta\tau}W^T_\alpha, \qquad [W^T_\tau, W^T_\sigma] = 0, \end{aligned} \tag{3.5}$$

for certain $\hat{\lambda}_{12}, \hat{\lambda}_{\alpha\beta}, \hat{\lambda}_\tau \in \mathcal{A}$ depending on $\hat{\lambda}, \hat{\lambda}_1, \hat{\lambda}_2 \in \mathcal{A}$ as given in [22].

4. Seiberg-Witten map

There are two kinds of Lorentz transformations for a field theory on θ-deformed space-time [23]: *observer Lorentz transformations* refer to passing to another reference frame; the physical action has to be invariant under such a tranformation. *Particle Lorentz transformations* refer to a repositioning of all particles in a given reference frame in which the background field θ remains unchanged. In general the physical action is not invariant under such a transformation. Since the particle Lorentz symmetry breaking must in principle be observable, it has to be gauge-invariant.

At first sight the particle Lorentz rotation is given by $W^R_{\hat{A}+\hat{\psi};\alpha\beta}$ defined in (3.3a). However, that transformation, applied to (3.1), does not lead to a gauge-invariant particle Lorentz symmetry breaking. This means that one rather has to split (3.3) in the following way [22]:

$$W^R_{\alpha\beta} = \tilde{W}^R_{\hat{A}+\hat{\psi};\alpha\beta} + \tilde{W}^R_{\theta;\alpha\beta}\,, \tag{4.1}$$

$$\tilde{W}^R_{\hat{A}+\hat{\psi};\alpha\beta} = W^R_{\hat{A}+\hat{\psi};\alpha\beta} - W^R_{\theta;\alpha\beta}(\theta^{\rho\sigma})\left(\left\langle \frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}, \frac{\delta}{\delta\hat{A}_\mu}\right\rangle + \left\langle \frac{d\hat{\psi}}{d\theta^{\rho\sigma}}, \frac{\delta}{\delta\hat{\psi}}\right\rangle\right), \tag{4.1a}$$

$$\tilde{W}^R_{\theta;\alpha\beta} = W^R_{\theta;\alpha\beta} + W^R_{\theta;\alpha\beta}(\theta^{\rho\sigma})\left(\left\langle \frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}, \frac{\delta}{\delta\hat{A}_\mu}\right\rangle + \left\langle \frac{d\hat{\psi}}{d\theta^{\rho\sigma}}, \frac{\delta}{\delta\hat{\psi}}\right\rangle\right). \tag{4.1b}$$

The transformation (4.1a) is then a particle Lorentz rotation if $\tilde{W}^R_{\hat{A}+\hat{\psi}}$ applied to the action (3.1) is gauge-invariant. This condition is solved by [22, 29]

$$\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}} = -\frac{1}{8}\big\{\hat{A}_\rho, \partial_\sigma\hat{A}_\mu + \hat{F}_{\sigma\mu}\big\}_\star + \frac{1}{8}\big\{\hat{A}_\sigma, \partial_\rho\hat{A}_\mu + \hat{F}_{\rho\mu}\big\}_\star + \hat{\Omega}_{\rho\sigma\mu}\,, \tag{4.2a}$$

$$\frac{d\hat{\psi}}{d\theta^{\rho\sigma}} = -\frac{1}{8}\hat{A}_\rho \star \big(\partial_\sigma\hat{\psi} + \hat{\hat{D}}_\sigma\hat{\psi}\big) + \frac{1}{8}\hat{A}_\sigma \star \big(\partial_\rho\hat{\psi} + \hat{\hat{D}}_\rho\hat{\psi}\big) + \hat{\Psi}_{\rho\sigma}\,, \tag{4.2b}$$

where $\hat{\Omega}_{\rho\sigma\mu}$ and $\hat{\Psi}_{\rho\sigma}$ transform covariantly under gauge transformations:

$$W^G_{\hat{\lambda}}\hat{\Omega}_{\rho\sigma\mu} = \mathrm{i}\big[\hat{\lambda}, \hat{\Omega}_{\rho\sigma\mu}\big]_\star\,, \qquad W^G_{\hat{\lambda}}\hat{\Psi}_{\rho\sigma} = \mathrm{i}\hat{\lambda}\star\hat{\Psi}_{\rho\sigma}\,. \tag{4.3a}$$

Compatibility in (4.2) requires

$$\dim\big(\hat{\Omega}_{\rho\sigma\mu}\big) = 3\,, \qquad \dim\big(\hat{\Psi}_{\rho\sigma}\big) = \frac{7}{2}\,, \qquad \hat{\Omega}_{\rho\sigma\mu} = \big(\hat{\Omega}_{\rho\sigma\mu}\big)^*\,. \tag{4.3b}$$

The relations (4.2) can now be regarded as first-order (Seiberg-Witten [1]) differential equations for the non-commutative fields $\hat{A}_\mu, \hat{\psi}$. As such they are solved by a power series in θ and the initial values A_μ, ψ for $\hat{A}_\mu, \hat{\psi}$, respectively, at $\theta = 0$. Inserting this solution into the action (3.1) one obtaines an action $\Sigma[A_\mu, \psi, \theta]$ which at each order n in θ is invariant under commutative gauge transformations and commutative observer Lorentz transformations [22]:

$$W^G_\lambda = \left\langle (\partial_\mu\lambda - \mathrm{i}[A_\mu, \lambda]), \frac{\delta}{\delta A_\mu}\right\rangle + \left\langle \mathrm{i}\lambda\psi, \frac{\delta}{\delta\psi}\right\rangle, \tag{4.4a}$$

$$\begin{aligned} W^R_{\alpha\beta} = {} & \left\langle (x_\alpha\partial_\beta A_\mu - x_\beta\partial_\alpha A_\mu + g_{\alpha\mu}A_\beta - g_{\beta\mu}A_\alpha)\,, \frac{\delta}{\delta A_\mu}\right\rangle + \\ & + \left\langle \left(x_\alpha\partial_\beta\psi - x_\beta\partial_\alpha\psi + \frac{1}{4}[\gamma_\alpha, \gamma_\beta]\psi\right), \frac{\delta}{\delta\psi}\right\rangle + \\ & + \left\langle (\delta^\rho_\alpha\theta_\beta{}^\sigma + \delta^\sigma_\alpha\theta^\rho{}_\beta - \delta^\rho_\beta\theta_\alpha{}^\sigma - \delta^\sigma_\beta\theta^\rho{}_\alpha), \frac{\delta}{\delta\theta^{\rho\sigma}}\right\rangle, \end{aligned} \tag{4.4b}$$

$$W^T_\tau = \left\langle \partial_\tau A_\mu, \frac{\delta}{\delta A_\mu}\right\rangle + \left\langle \partial_\tau\psi, \frac{\delta}{\delta\psi}\right\rangle. \tag{4.4c}$$

5. Non-commutative Yang-Mills-Dirac action to first order in θ

The solution of (4.2) is up to first order in θ given by

$$\hat{A}_\mu = A_\mu + \theta^{\alpha\beta}\Big(-\frac{1}{2}A_\alpha(\partial_\beta A_\mu + F_{\beta\mu})\Big) + \mathcal{O}(\theta^2)\,, \tag{5.1a}$$

$$\begin{aligned}\hat{\psi} = \psi + \theta^{\alpha\beta}\Big(&-\frac{1}{2}A_\alpha\partial_\beta\psi + \frac{1}{4}\partial_\alpha A_\beta\psi + \kappa_1 F_{\alpha\beta}\psi + \kappa_2 F_{\nu\beta}\gamma^\nu{}_\alpha\psi + \kappa_3 F_{\nu\rho}\gamma^{\nu\rho}{}_{\alpha\beta}\psi + \\ &+ \mathrm{i}\kappa_4 g^{\nu\rho}\gamma_{\alpha\beta}D_\nu D_\rho\psi + \mathrm{i}\kappa_5\gamma^\nu{}_\alpha D_\beta D_\nu\psi + \kappa_6 m\gamma_\alpha D_\beta\psi + \kappa_7 m\gamma^\nu{}_{\alpha\beta}D_\nu\psi + \\ &+ \mathrm{i}\kappa_8 m^2\gamma_{\alpha\beta}\psi\Big) + \mathcal{O}(\theta^2)\,,\end{aligned} \tag{5.1b}$$

where $D_\mu\psi = \partial_\mu\psi - \mathrm{i}A_\mu\psi$. The κ_i parametrize the solutions of (4.3) for[7] $\hat{\Omega}_{\rho\mu\sigma}$ and $\hat{\Psi}_{\rho\sigma}$. They play the role of additional coupling constants which parametrize (unphysical) *field redefinitions*, see [21] for the model without fermions.[8] Thus we expect these coupling constants to be power series in Planck's constant $\hbar$ in order to absorb possible divergences of the effective quantum action.[9] It turns out that all possible divergences in first order in θ are purely imaginary in momentum space. Hence we have written down only solutions of (4.3) which are purely imaginary in momentum space. In particular, the κ_i are real. We have introduced the completely antisymmetric gamma matrices

$$\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu,\gamma^\nu]\,, \qquad \gamma^{\mu\nu\rho} = \frac{1}{4}\{\gamma^\mu,[\gamma^\nu,\gamma^\rho]\}\,, \qquad \gamma^{\mu\nu\rho\sigma} = \frac{1}{8}[\gamma^\mu,\{\gamma^\nu,[\gamma^\rho,\gamma^\sigma]\}]\,, \tag{5.2a}$$

fulfilling

$$\gamma^\mu\gamma^{\nu_1\dots\nu_n} - (-1)^n\gamma^{\nu_1\dots\nu_n}\gamma^\mu = \sum_{j=1}^n(-1)^{j+1}\,2g^{\mu\nu_j}\,\gamma^{\nu_1\dots\nu_{j-1}\nu_{j+1}\dots\nu_j}\,, \qquad n=1,\dots,4\,. \tag{5.2b}$$

Note that $\hat{\Omega}_{\rho\sigma\mu}\in\mathcal{A}$ cannot contain a term bilinear in $\hat{\psi}$ because — within our framework presented in section 2 — there is no way to make an element of $\mathcal{A}$ out of two elements of $\mathcal{H}$. We shall see (this is one of the main results of the paper) that the lack of such a possibility makes the θ-expanded non-commutative QED unrenormalizable. Therefore some readers may suggest to enlarge somehow the framework of section 2 and to replace (5.1a) by

$$\begin{aligned}\hat{A}_\mu(x) = A_\mu(x) + \theta^{\alpha\beta}\Bigg(&-\frac{1}{2}A_\alpha(x)\big(\partial_\beta A_\mu(x) + F_{\beta\mu}(x)\big) \\ &+\mathrm{i}g^2\kappa_9\sum_{s,s'=1}^4\overline{\psi_s(x)}\big(\gamma^0\gamma_{\mu\alpha\beta}\big)^{ss'}\psi_{s'}(x)\Bigg) + \mathcal{O}(\theta^2)\,.\end{aligned} \tag{5.1a'}$$

[7] It is convenient to shift κ_1 by $1/8$ in order to add in (5.1b) the formally κ-independent term $\theta^{\alpha\beta}\partial_\alpha A_\beta\psi = \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta}\psi$ to the $\hat{\Psi}$-independent part of (4.2b).

[8] That the ambiguity in the solution of the Seiberg-Witten differential equation is linked to field redefinitions was already noticed in [30].

[9] Note that the coefficients in front of the non-covariant field monomials in (5.1) cannot be renormalized because they are fixed by gauge-invariance of the θ-expanded action.

To satisfy those readers we will indeed work with (5.1a'). We will see, however, that (5.1a') does not improve the result at all. Therefore, future work on similar subjects can stay withing the mathematical framework of section 2 from the very beginning.

Inserting (5.1) into the non-commutative Yang-Mills-Dirac action (3.1) we obtain[10] to first order in θ

$$\begin{aligned}
\Sigma_{c\ell} = \int d^4x \, \Big[& -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \bar\psi\big(\gamma^\mu(\mathrm{i}\partial_\mu + A_\mu) - m\big)\psi + \\
& + \theta^{\alpha\beta} \Big(-\frac{1}{2g^2} F_{\alpha\mu} F_{\beta\nu} F^{\mu\nu} + \frac{1}{8g^2} F_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} + \\
& \qquad + \Big(\frac{1}{4} - \kappa_2\Big) \mathrm{i}\bar\psi\gamma^\mu (2F_{\alpha\mu} D_\beta + \partial_\beta F_{\alpha\mu})\psi + \\
& \qquad + \Big(\kappa_1 - \frac{1}{2}\kappa_5\Big) \mathrm{i}\bar\psi\gamma^\mu (2F_{\alpha\beta} D_\mu + \partial_\mu F_{\alpha\beta})\psi + \\
& \qquad + (\kappa_2 + 2\kappa_4)\mathrm{i}\bar\psi\gamma_\alpha (2F^\mu{}_\beta D_\mu + \partial_\mu F^\mu{}_\beta)\psi + \\
& \qquad + \kappa_4 \bar\psi\gamma^\mu{}_{\alpha\beta}(-2D^\nu D_\nu D_\mu + 2\mathrm{i}F_\mu{}^\nu D_\nu\psi + \mathrm{i}\partial^\nu F_{\mu\nu})\psi - \\
& \qquad - \Big(\frac{1}{2}\kappa_2 + 2\kappa_3\Big) \mathrm{i}\bar\psi\gamma^{\mu\nu}{}_\alpha \partial_\beta F_{\mu\nu}\psi + \mathrm{i}(2\kappa_3 + \kappa_9)\bar\psi\gamma^\nu{}_{\alpha\beta}\partial^\mu F_{\mu\nu}\psi + \\
& \qquad + (-2\kappa_1 + \kappa_6) m\bar\psi F_{\alpha\beta}\psi + (-2\kappa_3 + \kappa_7) m\bar\psi F_{\nu\rho}\gamma^{\nu\rho}{}_{\alpha\beta}\psi + \\
& \qquad + (-2\kappa_4 + 2\kappa_7)\mathrm{i}m\bar\psi\gamma_{\alpha\beta} D^\nu D_\nu \psi + \\
& \qquad + (-\kappa_5 + \kappa_6 + 2\kappa_7)\mathrm{i}m\bar\psi\gamma^\nu{}_\alpha (2D_\nu D_\beta + \mathrm{i}F_{\nu\beta})\psi - \\
& \qquad - (2\kappa_7 + 2\kappa_8) m^2 \bar\psi\gamma^\nu{}_{\alpha\beta} D_\nu\psi - 2\mathrm{i}\kappa_8 m^3 \bar\psi\gamma_{\alpha\beta}\psi + \\
& \qquad + \mathrm{i}g^2\kappa_9 (\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_{\mu\alpha\beta}\psi)\Big)\Big] + \mathcal{O}(\theta^2)\,.
\end{aligned} \tag{5.3}$$

Note that $\{\kappa_1, \kappa_5, \kappa_6\}$ occur in (5.3) in the combinations $2\kappa_1 - \kappa_5$ and $\kappa_5 - \kappa_6$ only. In the massless case $m = 0$ there is only the combination $2\kappa_1 - \kappa_5$ suggesting not to eliminate κ_6. It is therefore no restriction to put

$$\kappa_5 = 0\,.$$

In order to pass to quantum field theory we have to add a BRST-invariant gauge-fixing term. This can be done before or after the Seiberg-Witten map [19]. We choose the more economical way of a gauge-fixing of (5.3):

$$\Sigma_{gf} = \int d^4x \, \Big(-\bar c \partial^\mu \partial_\mu c + B\partial^\mu A_\mu + \frac{\alpha}{2} B^2 \Big) = \int d^4x \; s\Big(\bar c\Big(\partial^\mu A_\mu + \frac{\alpha}{2} B\Big)\Big)\,, \tag{5.4a}$$

$$sA_\mu = \partial_\mu c\,, \qquad sc = 0\,, \qquad s\bar c = B\,, \qquad sB = 0\,, \qquad s\psi = \mathrm{i}c\psi\,, \qquad s\bar\psi = -\mathrm{i}\bar\psi c\,. \tag{5.4b}$$

Here one has to take into account that $\bar c, c$ and s are Grassmann-valued. Note that there are no interactions involving the ghosts $c, \bar c$ or the multiplier field B. Thus, the nilpotent

[10] We would like to draw the attention of the reader to the following typographical subtlety. From now on we will write down our formulae in terms of the *adjoint spinor* $\bar\psi = \psi^\dagger\gamma^0$ (a row of four elements of $L^2(\mathbb{R}^4)$). There is no risk to confuse the adjoint spinor with the previous operation of complex conjugation $\overline{\psi_s(x)}$ (for one element of $L^2(\mathbb{R}^4)$). The line symbolizing complex conjugation is longer. Additionally we restrict the number field we work with from $\mathbb{C}$ to $\mathbb{R}$ so that spinor ψ and its adjoint $\bar\psi$ must be regarded as independent.

BRST transformations (5.4b) are *linear* transformations of interacting fields. There is therefore no need to introduce sources (antifields) for the BRST transformations, and the BRST invariance of the total action $\Sigma = \Sigma_{cl} + \Sigma_{gf}$ can then be written as follows:

$$s\Sigma = \Big\langle \partial_\mu c, \frac{\delta\Sigma}{\delta A_\mu} \Big\rangle + \Big\langle -\mathrm{i}\bar{\psi} c, \frac{\delta\Sigma}{\delta\bar{\psi}} \Big\rangle + \Big\langle \mathrm{i}c\psi, \frac{\delta\Sigma}{\delta\psi} \Big\rangle + \Big\langle B, \frac{\delta\Sigma}{\delta\bar{c}} \Big\rangle = 0\,. \tag{5.5}$$

6. Feynman rules

We denote by $\Gamma[\phi_{i,c\ell}]$ the generating functional of one-particle irreducible (1PI) Green's functions and by $Z_c[J_i]$ the generating functional for connected Green's functions. Both are related by Legendre transformation

$$Z_c[J_i] = \Gamma[\phi_{i,c\ell}] + \sum_i \int d^4x\ \phi_{i,c\ell}(x)\, J_i(x)\,, \qquad J_i(x) = -\Big\langle \delta^4(x{-}y), \frac{\delta\Gamma}{\delta\phi_{i,c\ell}(y)} \Big\rangle\,. \tag{6.1}$$

Switching to momentum space,[11] 1PI Green's functions are obtained by functional derivation:

$$(2\pi)^4\delta^4(p_1 + \dots + p_n)\Gamma_{\phi_1\dots\phi_n}(p_1,\dots,p_n) = \frac{\delta^n\Gamma[\phi_{i,c\ell}]}{\delta\phi_{1,c\ell}(p_1)\dots\delta\phi_{n,c\ell}(p_n)}\Big|_{\phi_{i,c\ell}=0}\,. \tag{6.2}$$

Functional derivation in momentum space is defined by

$$\frac{\delta\phi_{i,c\ell}(p_i)}{\delta\phi_{j,c\ell}(p_j)} = \delta_i^j\ (2\pi)^4\delta^4(p_i{-}p_j)\,. \tag{6.3}$$

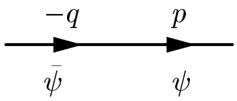
A parity factor of -1 has to be inserted for each commutation of a Grassmann-valued derivative operator with a Grassmann-valued field. Similarly, connected Green's functions are obtained by

$$(2\pi)^4\delta^4(p_1 + \dots + p_n)\Delta^{\phi_1\dots\phi_n}(p_1,\dots,p_n) = \frac{\delta^n Z_c[J_i]}{\delta J_1(p_1)\dots\delta J_n(p_n)}\Big|_{J_i=0}\,. \tag{6.4}$$

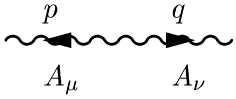
We introduce a bigrading (τ,ℓ) for all Green's functions, with τ being the number of factors of θ and ℓ the number of loops. The $(\ell{=}0)$-part corresponds to taking for Γ the action $\Sigma = \Sigma_{c\ell} + \Sigma_{gf}$.

6.1 Propagators

Feynman rules for propagators are obtained from the bilinear $(\tau{=}0, \ell{=}0)$-part[12] of Z_c. We only need the propagators for the sources of photons and electrons:



$$\Delta^{\bar{\psi}\psi}_{(0,0)}(q,p) = -\frac{\gamma^\mu p_\mu + m}{p^2 - m^2 + \mathrm{i}\epsilon}\,, \tag{6.5a}$$



$$\Delta^{AA}_{(0,0)\mu\nu}(p,q) = \frac{g^2}{p^2 + \mathrm{i}\epsilon}\left(g_{\mu\nu} - \Big(1{-}\frac{\alpha}{g^2}\Big)\frac{p_\mu p_\nu}{p^2 + \mathrm{i}\epsilon}\right)\,. \tag{6.5b}$$

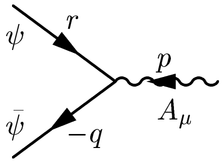
The ghost propagator and the mixed A-B propagator are not required because there are no vertices involving B and ghosts.

[11] Our Fourier conventions are $f(x) = \int \frac{d^4k}{(2\pi)^4}\ \mathrm{e}^{-\mathrm{i}k\cdot x}\ \tilde{f}(k)$ and $\tilde{f}(p) = \int d^4x\ \mathrm{e}^{\mathrm{i}k\cdot x}\ f(x)$.

[12] We regard θ-dependent terms which are bilinear in fields as vertices.

6.2 Vertices independent of θ

Feynman rules for vertices are obtained from the interaction (ℓ=0)-part of Γ, i.e. from the interaction part of the total action Σ. The only vertex which is independent of θ is the standard QED vertex:



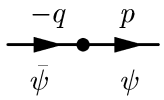
$$\Gamma^{(0,0)\mu}_{A\bar{\psi}\psi}(p,q,r) = \gamma^\mu \,. \tag{6.6}$$

The free part of the action Σ leads to the following Green's functions, which by definition do not give rise to Feynman rules:

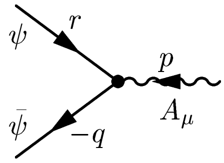
$$\begin{aligned}
\Gamma^{(0,0)\mu\nu}_{AA}(p,q) &= -\frac{1}{g^2}\big(p^2 g^{\mu\nu} - p^\mu p^\nu\big)\,, \qquad \Gamma^{(0,0)}_{\bar{\psi}\psi}(q,p) = p_\mu\gamma^\mu - m\,,\\
\Gamma^{(0,0)\mu}_{AB}(p,q) &= -\mathrm{i}p^\mu\,, \qquad \Gamma^{(0,0)}_{BB}(p,q) = \alpha\,, \qquad \Gamma^{(0,0)}_{\bar{c}c}(q,p) = p^2\,.
\end{aligned} \tag{6.7}$$

6.3 Vertices linear in θ

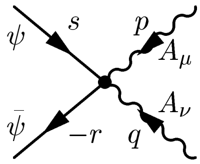
At first order in θ we have the following graphs:



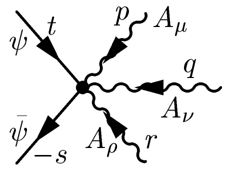
$\Gamma^{(1,0)}_{\bar{\psi}\psi}(p)$



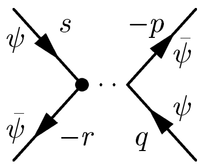
$\Gamma^{(1,0)\mu}_{A\bar{\psi}\psi}(p,q,r)$



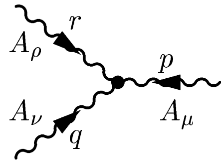
$\Gamma^{(1,0)\mu\nu}_{AA\bar{\psi}\psi}(p,q,r,s)$



$\Gamma^{(1,0)\mu\nu\rho}_{AAA\bar{\psi}\psi}(p,q,r,s,t)$



$\Gamma^{(1,0)}_{\bar{\psi}\psi;\bar{\psi}\psi}(p,q;r,s)$



$$\Gamma^{(1,0)\mu\nu\rho}_{AAA;1+2}(p,q,r) \equiv \Gamma^{(1,0)\mu\nu\rho}_{AAA;1}(p,q,r) + \Gamma^{(1,0)\mu\nu\rho}_{AAA;2}(p,q,r)\,, \tag{6.8}$$

where

$$\begin{aligned}
\Gamma^{(1,0)}_{\bar{\psi}\psi}(q,p) = 2\mathrm{i}\theta_{\alpha\beta}\Big(&-\kappa_8 m^3\gamma^{\alpha\beta} + (\kappa_4{-}\kappa_7)mp^2\gamma^{\alpha\beta} - m(\kappa_6{+}2\kappa_7)p_\mu p^\beta\gamma^{\mu\alpha} +\\
&+ (\kappa_7{+}\kappa_8)m^2 p_\mu\gamma^{\mu\alpha\beta} - \kappa_4 p_\mu p^2\gamma^{\mu\alpha\beta}\Big)\,,
\end{aligned} \tag{6.9a}$$

$$\begin{aligned}
\Gamma^{(1,0)\mu}_{A\bar{\psi}\psi}(p,q,r) = \mathrm{i}\theta_{\alpha\beta}\Bigg(&\left(\left(2\kappa_2{-}\frac{1}{2}\right)p^\alpha r^\beta\gamma^\mu - \left(2\kappa_2{-}\frac{1}{2}\right)p^\nu r^\beta\gamma_\nu g^{\mu\alpha} +\right.\\
&+ \left(\frac{1}{4}{+}2\kappa_1{-}\kappa_2\right)p^\nu p^\beta\gamma_\nu g^{\mu\alpha} + (4\kappa_1)\gamma_\nu r^\nu p^\beta g^{\mu\alpha} +\\
&+ (2\kappa_6{-}4\kappa_1)mg^{\mu\alpha}p^\beta + (\kappa_2{+}2\kappa_4)(p^2+2pr)\gamma^\beta g^{\mu\alpha} -\\
&- (\kappa_2{+}2\kappa_4)(p^\mu+2r^\mu)r^\alpha\gamma^\beta - (\kappa_2{+}4\kappa_3)\gamma^{\mu\nu\alpha}p_\nu p^\beta -\\
&- (\kappa_6{+}2\kappa_7)m(p^\beta+2r^\beta)\gamma^{\mu\alpha} + (\kappa_6{+}2\kappa_7)m(p_\nu+2r_\nu)\gamma^{\nu\beta}g^{\mu\alpha} +
\end{aligned}$$

$$
\begin{aligned}
&+\ (2\kappa_4-2\kappa_7)m(p^\mu+2r^\mu)\gamma^{\alpha\beta}+(2\kappa_7-4\kappa_3)mp_\nu\gamma^{\mu\nu\alpha\beta}+\\
&+\ (2\kappa_3+\kappa_9-\kappa_4)p^\mu p_\nu\gamma^{\nu\alpha\beta}-(2\kappa_3+\kappa_9-\kappa_4)p^2\gamma^{\mu\alpha\beta}-\\
&-\ 2\kappa_4(p^2+pr+r^2)\gamma^{\mu\alpha\beta}-2\kappa_4(p^\mu r_\nu+r^\mu p_\nu+2r^\mu r_\nu)\gamma^{\nu\alpha\beta}+\\
&+\ (2\kappa_7+2\kappa_8)m^2\gamma^{\mu\alpha\beta}\Big)\,,
\end{aligned}
\tag{6.9b}
$$

$$
\begin{aligned}
\Gamma^{(1,0)\mu\nu}_{AA\bar\psi\psi}(p,q,r,s)=\mathrm{i}\theta_{\alpha\beta}\bigg(&\left(\frac{1}{2}-2\kappa_2\right)(\gamma^\mu g^{\nu\alpha}p^\beta+\gamma^\nu g^{\mu\alpha}q^\beta)+\\
&+\left(\frac{1}{2}-2\kappa_2\right)\gamma_\rho(p^\rho g^{\alpha\mu}g^{\beta\nu}-q^\rho g^{\alpha\mu}g^{\beta\nu})+\\
&+\ 4\kappa_1(g^{\nu\alpha}q^\beta\gamma^\mu+g^{\mu\alpha}p^\beta\gamma^\nu)+(2\kappa_2+4\kappa_4)g^{\mu\nu}\gamma^\alpha(p^\beta+q^\beta)+\\
&+\ (2\kappa_2+4\kappa_4)\gamma^\beta(g^{\mu\alpha}p^\nu+g^{\nu\alpha}q^\mu)-2\kappa_4\gamma^{\mu\alpha\beta}(p^\nu+q^\nu+2s^\nu)-\\
&-\ 2\kappa_4\gamma^{\nu\alpha\beta}(p^\mu+q^\mu+2s^\mu)-2\kappa_4 g^{\mu\nu}\gamma^{\rho\alpha\beta}(p_\rho+q_\rho+2s_\rho)+\\
&+\ (4\kappa_7+2\kappa_6)m(\gamma^{\mu\beta}g^{\nu\alpha}+\gamma^{\nu\beta}g^{\mu\alpha})+\\
&+(4\kappa_4-4\kappa_7)mg^{\mu\nu}\gamma^{\alpha\beta}\bigg)\,,
\end{aligned}
\tag{6.9c}
$$

$$
\Gamma^{(1,0)\mu\nu\rho}_{AAA\bar\psi\psi}(p,q,r,s,t)=-4\mathrm{i}\theta_{\alpha\beta}\kappa_4\big(g^{\mu\nu}\gamma^{\rho\alpha\beta}+g^{\nu\rho}\gamma^{\mu\alpha\beta}+g^{\rho\mu}\gamma^{\nu\alpha\beta}\big)\,,
\tag{6.9d}
$$

$$
\Gamma^{(1,0)}_{\bar\psi\psi;\bar\psi\psi}(p,q;r,s)=\mathrm{i}g^2\theta_{\alpha\beta}\kappa_9\gamma_\mu\otimes\gamma^{\mu\alpha\beta}\,,
\tag{6.9e}
$$

$$
\begin{aligned}
\Gamma^{(1,0)\mu\nu\rho}_{AAA;1}(p,q,r)=\frac{1}{g^2}(1+\lambda_1)\mathrm{i}\theta_{\alpha\beta}\Big(&g^{\alpha\mu}g^{\beta\nu}\big((qr)p^\rho-(pr)q^\rho\big)+g^{\alpha\nu}g^{\beta\rho}\big((rp)q^\mu-(qp)r^\mu\big)+\\
&+\ g^{\alpha\rho}g^{\beta\mu}\big((pq)r^\nu-(rq)p^\nu\big)+\\
&+\ g^{\alpha\mu}\big((g^{\nu\rho}(pq)-p^\nu q^\rho)r^\beta+(g^{\nu\rho}(rp)-r^\nu p^\rho)q^\beta\big)+\\
&+\ g^{\alpha\nu}\big((g^{\rho\mu}(qr)-q^\rho r^\mu)p^\beta+(g^{\rho\mu}(pq)-p^\rho q^\mu)r^\beta\big)+\\
&+\ g^{\alpha\rho}\big((g^{\mu\nu}(rp)-r^\mu p^\nu)q^\beta+(g^{\mu\nu}(qr)-q^\mu r^\nu)p^\beta\big)+\\
&+\ g^{\mu\nu}\big(p^\rho q^\alpha r^\beta+q^\rho p^\alpha r^\beta\big)+g^{\nu\rho}\big(q^\mu r^\alpha p^\beta+r^\mu q^\alpha p^\beta\big)+\\
&+\ g^{\rho\mu}\big(r^\nu p^\alpha q^\beta+p^\nu r^\alpha q^\beta\big)\Big),
\end{aligned}
\tag{6.9f}
$$

$$
\begin{aligned}
\Gamma^{(1,0)\mu\nu\rho}_{AAA;2}(p,q,r)=\frac{1}{g^2}(1+\lambda_2)\mathrm{i}\theta_{\alpha\beta}\Big(&-g^{\alpha\mu}(g^{\nu\rho}(rq)-r^\nu q^\rho)p^\beta-g^{\alpha\nu}(g^{\rho\mu}(pr)-p^\rho r^\mu)q^\beta-\\
&-\ g^{\alpha\rho}(g^{\mu\nu}(qp)-q^\mu p^\nu)r^\beta\Big)\,.
\end{aligned}
\tag{6.9g}
$$

The last two terms (6.9f), (6.9g) occur as a sum with $\lambda_1=\lambda_2=0$, we split them artificially in order to discuss possible extensions

$$
\mathrm{YM}=-\frac{1}{4g^2}\int\left(\hat F_{\mu\nu}\star\hat F^{\mu\nu}+2\lambda_1\theta^{\alpha\beta}\hat F_{\alpha\mu}\star\hat F_{\beta\nu}\star\hat F^{\mu\nu}-\frac{1}{2}\lambda_2\theta^{\alpha\beta}\hat F_{\alpha\beta}\star\hat F_{\mu\nu}\star\hat F^{\mu\nu}\right)
\tag{6.10}
$$

to the bosonic action. These Feynman rules are subject to momentum conservation $p+q=0$, $p+q+r=0$, $p+q+r+s=0$ and $p+q+r+s+t=0$, respectively. The strange fifth graph in (6.8) symbolizes a single vertex. The dotted line permits momentum exchange but does not affect the spinor structure.

6.4 Concatenation of propagators and vertices

To an 1PI Feynman graph one associates an integral by concatenation of propagators (inner lines only) and vertices, matching momenta and Lorentz indices and preserving the sense of the arrows. Concatenation from left to right of fermion propagators and vertices is always in opposite sense of the arrow, because the adjoint spinor symbolized by an outgoing arrow is on the left. To any closed loop with loop momentum k_i one associates an integration operator $\frac{\hbar}{\mathrm{i}} \int \frac{d^4 k_i}{(2\pi)^4}$. To each closed electron line one associates the operator $-\operatorname{tr}$, where tr denotes the trace over the γ-matrices in the loop. If the graph has an S-fold symmetry the integral has to be devided by S.

The integration over all internal loop momenta of a subgraph with N_A external photon lines and N_ψ external electron lines and with a total number of T factors of θ is expected to be ultraviolet divergent if

$$4 + 2T - N_A - \frac{3}{2} N_\psi \geq 0\,. \tag{6.11}$$

Note that N_ψ is always even. The problem is to make sense of these meaningless integrals in a way preserving locality.

7. One-loop Feynman graphs independent of θ

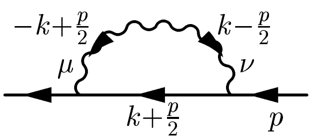
First we compute all divergent one-loop graphs built out of the single vertex (6.6) which is independent of θ. From (6.11) we see that the problematic graphs are those with

$$(N_\psi, N_A) \in \{\ (2,0)\,,\ (2,1)\,,\ (0,2)\,,\ (0,3)\,,\ (0,4)\ \}\,. \tag{7.1}$$

We employ analytic regularization in terms of a complex variable ε, $|\varepsilon| \to 0$, see appendix A for details. The advantage is that with analytic regularization we are on the safe side with respect to the algebra of γ-matrices. Moreover, we can — in the divergent part — arbitrarily shift the integration momentum and naively eliminate common factors in the numerator and denominator (verified for all integrals to evaluate).

For the electron selfenergy $(N_\psi, N_A) = (2,0)$ we obtain

$\Gamma^{(0,1)}_{\bar\psi\psi}(q,p) =$



$$\begin{aligned}
&= \Gamma^{(0,0)\mu}_{A\bar\psi\psi}\Big(-k+\frac{p}{2}, -p, k+\frac{p}{2}\Big)\, \Delta^{\bar\psi\psi}_{(0,0)}\Big(-k-\frac{p}{2}, k+\frac{p}{2}\Big)\, \Gamma^{(0,0)\nu}_{A\bar\psi\psi}\Big(k-\frac{p}{2}, -k-\frac{p}{2}, p\Big) \times \\
&\quad \times \Delta^{AA}_{(0,0)\mu\nu}\Big(-k+\frac{p}{2}, k-\frac{p}{2}\Big) \\
&= \frac{\hbar g^2}{(4\pi)^2 \varepsilon}\Big(\frac{\alpha}{g^2}\gamma^\rho p_\rho - \Big(3+\frac{\alpha}{g^2}\Big) m\Big) + \mathcal{O}(1) \\
&= \frac{\hbar g^2}{(4\pi)^2 \varepsilon}\Big(\frac{\alpha}{2g^2} N_\psi + 3m\frac{\partial}{\partial m}\Big)\Gamma^{(0,1)}_{\bar\psi\psi}(p) + \mathcal{O}(1)\,.
\end{aligned} \tag{7.2}$$

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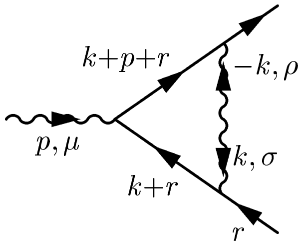
Next, the one-loop QED vertex correction $(N_\psi, N_A) = (2,1)$ is computed to

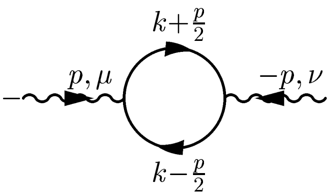
$$\Gamma^{(0,1)\mu}_{A\bar\psi\psi}(p,q,r) =$$



$$\begin{aligned}
&= \Delta^{AA}_{(0,0)\sigma\rho}(k,-k)\Gamma^{(0,0)\rho}_{A\bar\psi\psi}(-k,-p-r,k+p+r)\Delta^{\bar\psi\psi}_{(0,0)}(-k-p-r,k+p+r)\times \\
&\quad\times \Gamma^{(0,0)\mu}_{A\bar\psi\psi}(p,-k-p-r,k+r)\Delta^{\bar\psi\psi}_{(0,0)}(-k-r,k+r)\Gamma^{(0,0)\sigma}_{A\bar\psi\psi}(k,-k-r,r) \\
&= \frac{\hbar\alpha}{(4\pi)^2\varepsilon}\gamma^\mu + \mathcal{O}(1) \\
&= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(\frac{\alpha}{2g^2}N_\psi + 0\,N_A\right)\Gamma^{(0,0)\sigma}_{A\bar\psi\psi}(p,q,r) + \mathcal{O}(1)\,.
\end{aligned} \tag{7.3}$$

For the photon selfenergy $(N_\psi, N_A) = (0,2)$ we obtain

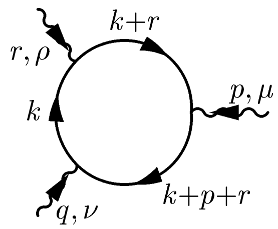
$$\Gamma^{(0,1)\mu\nu}_{AA}(p,q) = -$$



$$\begin{aligned}
&= -\operatorname{tr}\Big(\Gamma^{(0,0)\mu}_{A\bar\psi\psi}\Big(p,-k-\frac{p}{2},k-\frac{p}{2}\Big)\Delta^{\bar\psi\psi}_{(0,0)}\Big(-k+\frac{p}{2},k-\frac{p}{2}\Big)\times \\
&\qquad\times\Gamma^{(0,0)\nu}_{A\bar\psi\psi}\Big(-p,-k+\frac{p}{2},k+\frac{p}{2}\Big)\Delta^{\bar\psi\psi}_{(0,0)}\Big(-k-\frac{p}{2},k+\frac{p}{2}\Big)\Big) \\
&= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(-\frac{4}{3g^2}(p^2g^{\mu\nu} - p^\mu p^\nu)\right) + \mathcal{O}(1) \\
&= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(0\,N_A - \frac{4}{3}g^2\frac{\partial}{\partial g^2}\right)\Gamma^{AA}_{(0,0)}(p) + \mathcal{O}(1)\,.
\end{aligned} \tag{7.4}$$

The photon three- and four-point functions $(N_\psi, N_A) = (0,3)$ and $(N_\psi, N_A) = (0,4)$ are actually convergent due to gauge invariance and its preservation in analytic regularization. For instance, in the three-point function

$$\Gamma^{(0,1)\mu\nu\rho}_{AAA}(p,q,r) = \Bigg\{ -$$



$$\Bigg\} + \Big\{(q,\nu) \leftrightarrow (r,\rho)\Big\} = \mathcal{O}(1) \tag{7.5}$$

the divergent contributions of both graphs cancel (Furry's theorem).

In order to absorb the divergences (7.2), (7.3) and (7.4) the fields and parameters of the model must depend on $\hbar$ in the following way:

$$g = \left(1 + \frac{2}{3}\frac{\hbar g_0^2}{(4\pi)^2\varepsilon} + \mathcal{O}(\hbar^2)\right) g_0\,, \tag{7.6a}$$

$$m = \left(1 - 3\frac{\hbar g_0^2}{(4\pi)^2\varepsilon} + \mathcal{O}(\hbar^2)\right) m_0\,, \tag{7.6b}$$

$$\bar\psi = \left(1 - \frac{\alpha}{2g_0^2}\frac{\hbar g_0^2}{(4\pi)^2\varepsilon} + \mathcal{O}(\hbar^2)\right) \bar\psi_0\,, \tag{7.6c}$$

$$\psi = \left(1 - \frac{\alpha}{2g_0^2}\frac{\hbar g_0^2}{(4\pi)^2\varepsilon} + \mathcal{O}(\hbar^2)\right) \psi_0\,, \tag{7.6d}$$

$$A_\mu = (A_\mu)_0 + \mathcal{O}(\hbar^2)\,. \tag{7.6e}$$

8. One-loop Feynman graphs linear in θ

In this section we compute the divergent one-loop Green's functions involving a vertex linear in θ. Assuming a regularization scheme preserving gauge invariance (such as analytic regularization), we expect at order 1 in θ the following Hermitean counterterms of dimension[13] 0 which are purely imaginary in momentum space:

$$B_1 = \int d^4x\, \theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} F^{\mu\nu}\,, \qquad B_2 = \int d^4x\, \theta^{\alpha\beta} F_{\alpha\beta} F_{\mu\nu} F^{\mu\nu}\,, \tag{8.1a}$$

$$\Phi_0 = \int d^4x\, \mathrm{i}\theta^{\alpha\beta} \bar\psi \gamma^\mu (2F_{\alpha\mu} D_\beta + \partial_\beta F_{\alpha\mu})\psi\,, \tag{8.1b}$$

$$\begin{aligned}
\Phi_1 &= \int d^4x\, \mathrm{i}\theta^{\alpha\beta} \bar\psi \gamma^\mu (2F_{\alpha\beta} D_\mu + \partial_\mu F_{\alpha\beta})\psi\,, \qquad \Phi_2 = \int d^4x\, \mathrm{i}\theta^{\alpha\beta} \bar\psi \gamma_\alpha (2F_{\mu\beta} D^\mu + \partial^\mu F_{\mu\beta})\psi\,,\\
\Phi_3 &= \int d^4x\, \theta^{\alpha\beta} \bar\psi \gamma^{\mu\alpha\beta} (-2D^\nu D_\nu D_\mu + 2\mathrm{i}F_\mu{}^\nu D_\nu + \mathrm{i}\partial^\nu F_{\mu\nu})\psi\,,\\
\Phi_4 &= \int d^4x\, \mathrm{i}\theta^{\alpha\beta} \bar\psi \gamma_{\mu\nu\alpha} \partial_\beta F^{\mu\nu} \psi\,, \qquad \Phi_5 = \int d^4x\, \mathrm{i}\theta^{\alpha\beta} \bar\psi \gamma_{\mu\alpha\beta} \partial_\nu F^{\nu\mu} \psi\,,
\end{aligned} \tag{8.1c}$$

$$\tilde\Phi_1 = \int d^4x\, \mathrm{i}\theta^{\alpha\beta} (\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_{\mu\alpha\beta}\psi)\,, \tag{8.1d}$$

$$\begin{aligned}
M_1 &= \int d^4x\, \mathrm{i}\theta^{\alpha\beta} m^3 \bar\psi \gamma_{\alpha\beta} \psi\,, \qquad M_2 = \int d^4x\, \theta^{\alpha\beta} m^2 \bar\psi \gamma_{\mu\alpha\beta} D^\mu \psi\,,\\
M_3 &= \int d^4x\, \mathrm{i}\theta^{\alpha\beta} m \bar\psi \gamma^\mu{}_\alpha (2D_\mu D_\beta + \mathrm{i}F_{\mu\beta})\psi\,, \qquad M_4 = \int d^4x\, \mathrm{i}\theta^{\alpha\beta} m \bar\psi \gamma_{\alpha\beta} D^\nu D_\nu \psi\,,\\
M_5 &= \int d^4x\, \theta^{\alpha\beta} m \bar\psi \gamma_{\mu\nu\alpha\beta} F^{\mu\nu} \psi\,, \qquad M_6 = \int d^4x\, \theta^{\alpha\beta} m \bar\psi F_{\alpha\beta} \psi\,.
\end{aligned} \tag{8.1e}$$

Naive comparison of (5.3) with (8.1) shows immediately that the non-commutative Yang-Mills-Dirac action (3.1) expanded to first order in θ cannot expected to be renormalizable. Indeed, for the absorption of the five divergences corresponding to $\Phi_1 \cdots \Phi_5$ we only have the field redefinition parameters $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ at disposal, because κ_9 (if included) is already

[13] The power-counting dimensions dim are $\dim(d^4x) = -4$ and $\dim(\delta(x-y)) = 4$.

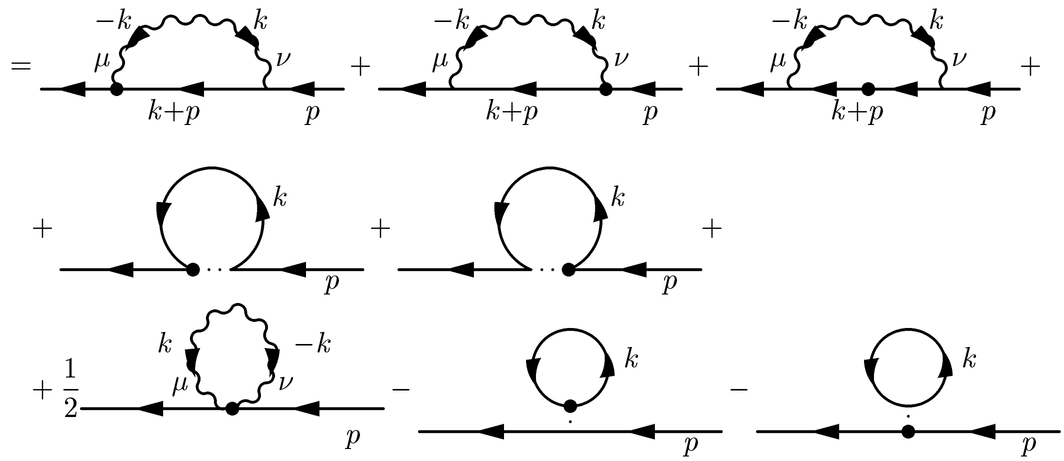
fixed by the divergence corresponding to $\tilde{\Phi}_1$. Even if we allow for a renormalization of θ to deal with Φ_0, there is at least one missing parameter to achieve renormalizability in the massless case without a symmetry. The massive case is worse: there are only three parameters $\kappa_6, \kappa_7, \kappa_8$ to absorb the six divergences corresponding to $M_1 \cdots M_6$. Finally, for the bosonic divergences corresponding to B_1, B_2 there is no parameter at all available (for $\lambda_1 = \lambda_2 = 0$ in (6.10)). Thus, unless there are symmetries, θ-expanded non-commutative QED cannot be renormalizable.[14]

For simplicity we choose from now on the Feynman gauge $\alpha = g_0^2 = g^2 + \mathcal{O}(\hbar)$.

8.1 Two electrons, no photon

We begin with the electron selfenergy, allowing for a renormalization of θ:

$$\Gamma^{(1,1)}_{\bar\psi\psi}(q,p) =$$



$$\begin{aligned}
&= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\Bigg(\frac{1}{2}N_\psi + 0\,N_A + 3m\frac{\partial}{\partial m} + \tau\theta^{\alpha\beta}\frac{\partial}{\partial\theta^{\alpha\beta}} + \\
&\quad + \left(\frac{1}{48} - \frac{1}{6}\kappa_1 + \frac{1}{2}\kappa_2 + \frac{2}{3}\kappa_3 + \left(-\tau - \frac{1}{2}\right)\kappa_4 + \frac{1}{3}\kappa_9\right)\frac{\partial}{\partial\kappa_4} + \\
&\quad + \left(\frac{1}{6} + \frac{2}{3}\kappa_1 - \kappa_2 - \frac{2}{3}\kappa_3 - 3\kappa_4 + (-\tau - 3)\kappa_6 + 2\kappa_7 - \frac{1}{3}\kappa_9\right)\frac{\partial}{\partial\kappa_6} + \\
&\quad + \left(-\frac{1}{2}\kappa_2 - \kappa_3 - \frac{3}{2}\kappa_4 + \frac{1}{2}\kappa_6 + (-\tau - 3)\kappa_7 + \frac{1}{2}\kappa_9\right)\frac{\partial}{\partial\kappa_7} + \\
&\quad + \left(\frac{1}{16} - \frac{1}{2}\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_6 + 2\kappa_7 + (-\tau - 6)\kappa_8 + \frac{1}{2}\kappa_9\right)\frac{\partial}{\partial\kappa_8}\Bigg) \times \\
&\quad \times \left(\Gamma^{(1,0)}_{\bar\psi\psi}(p)\right) + \mathcal{O}(1)\,.
\end{aligned} \tag{8.2}$$

This is indeed a renormalization of $\kappa_4, \kappa_6, \kappa_7, \kappa_8$!

[14] This refers to renormalizability of the θ-expansion of (3.1). One can always add to (3.1) on unexpanded level θ-dependent terms as in (6.10) in order to compensate the divergences of the type (8.1). However, such a theory looses all predictivity and hence would be regarded as "non renormalizable".

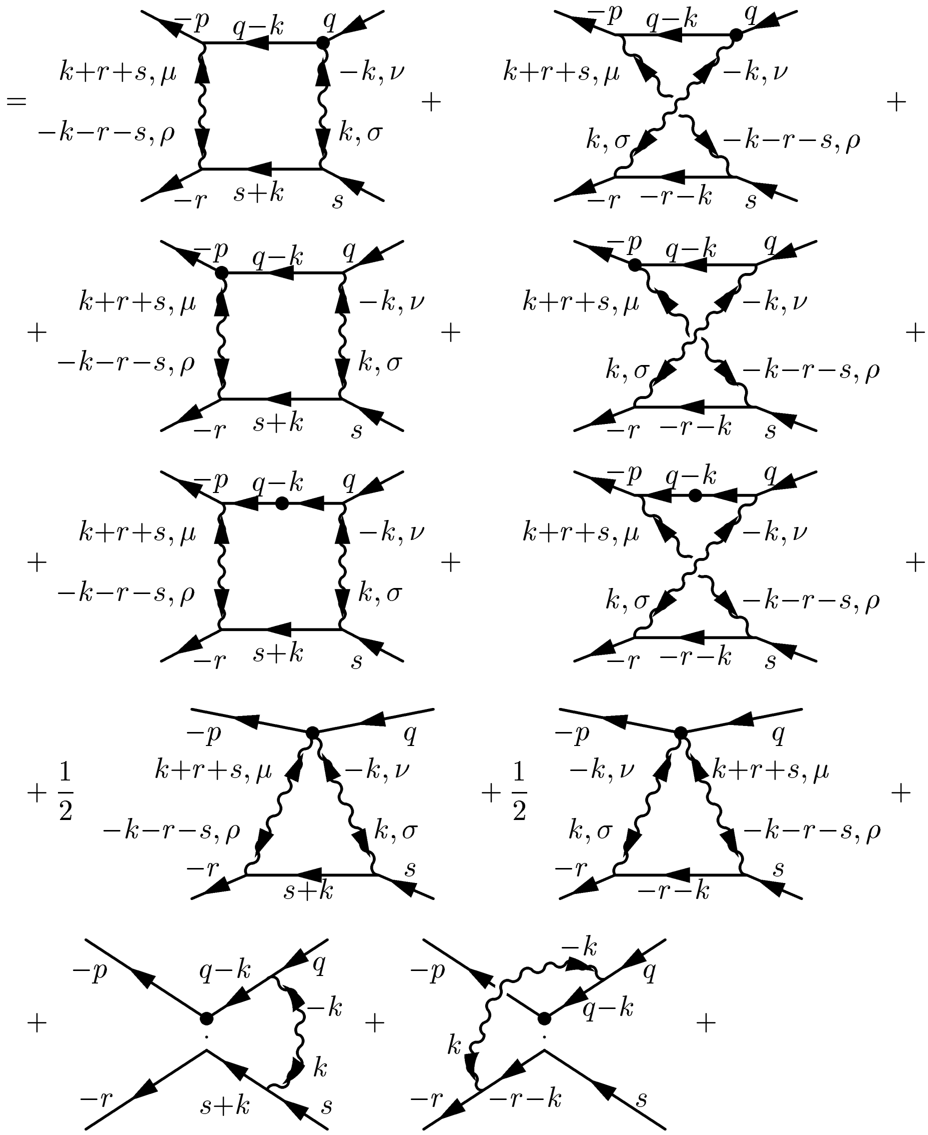
8.2 Four electrons, no photon

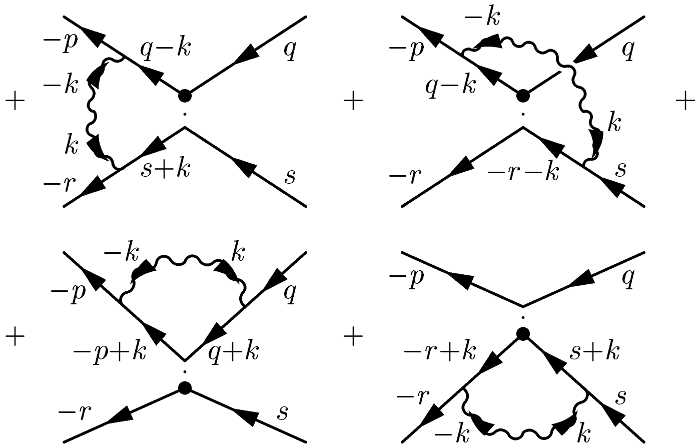
Next we consider one-loop corrections to the electron four-point function $\Gamma_{\bar\psi\psi;\bar\psi\psi}(p,q;r,s)$ in first order in θ. Since $\Gamma_{\bar\psi\psi;\bar\psi\psi}(p,q;r,s)$ is independent of external momenta and no further label distinguishes the two electron pairs in $\Gamma_{\bar\psi\psi;\bar\psi\psi}(p,q;r,s)$, we can consider $\Gamma_{\bar\psi\psi;\bar\psi\psi}(p,q;r,s)$ in the asymmetric form where the electrons labelled by p,q attach to γ^μ and those labelled by r,s attach to $\gamma_{\mu\alpha\beta}$ in (6.9e). This reduces the number of graphs to compute: we simply choose the external momenta of the divergent part of the one-loop graphs according to this prescription. All but the last two graphs below can then be arranged in pairs which contain the following concatenation of Feynman rules:

$$\gamma^\rho(-\gamma^\tau(s_\tau+k_\tau)-m)\gamma^\sigma+\gamma^\sigma(-\gamma^\tau(-r_\tau-k_\tau)-m)\gamma^\rho=2\gamma^{\rho\sigma\tau}k_\tau+\text{terms independent of }k\,.$$

Thus, these subgraphs correspond to the external momenta (r,s). We therefore have

$$\Gamma^{(1,1)}_{\bar\psi\psi;\bar\psi\psi}(p,q;r,s)=$$





$$= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(\frac{1}{2}N_\psi - \frac{4}{3}g^2\frac{\partial}{\partial g^2} + \tau\theta^{\alpha\beta}\frac{\partial}{\partial\theta^{\alpha\beta}} + \left(\frac{3}{4} + \left(-\tau + \frac{4}{3}\right)\kappa_9\right)\frac{\partial}{\partial\kappa_9}\right)\times$$
$$\times\,\Gamma^{(1,0)}_{\bar\psi\psi;\bar\psi\psi}(p,q;r,s) + \mathcal{O}(1)\,. \tag{8.3}$$

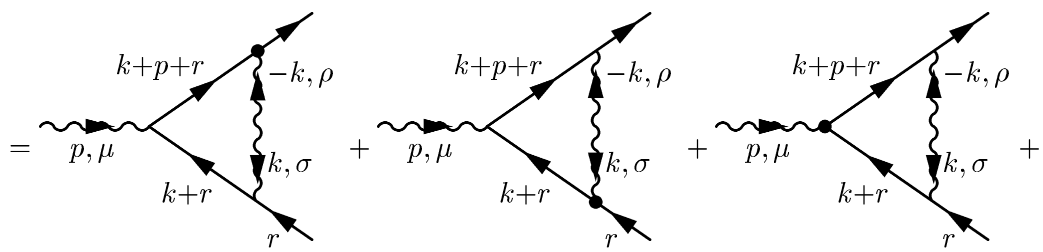
The divergence which is expressed by the factor 3/4 in (8.3) is problematic. We had mentioned in the discussion around (5.1a') that the mathematical framework of section 2 imposes $\kappa_9 = 0$ to all orders in $\hbar$. But this contradicts (8.3) which enforces an $\hbar$-renormalization of κ_9. That was the reason why we have included κ_9. Nevertheless it will turn out that the inclusion of κ_9 does not help. In order to make this transparent we introduce a switch ζ which according to (8.3) should take the value ζ=1 and for mathematical reasons the value ζ=0 (leaving us with an unrenormalizable divergence in the electron four-point function). We therefore write

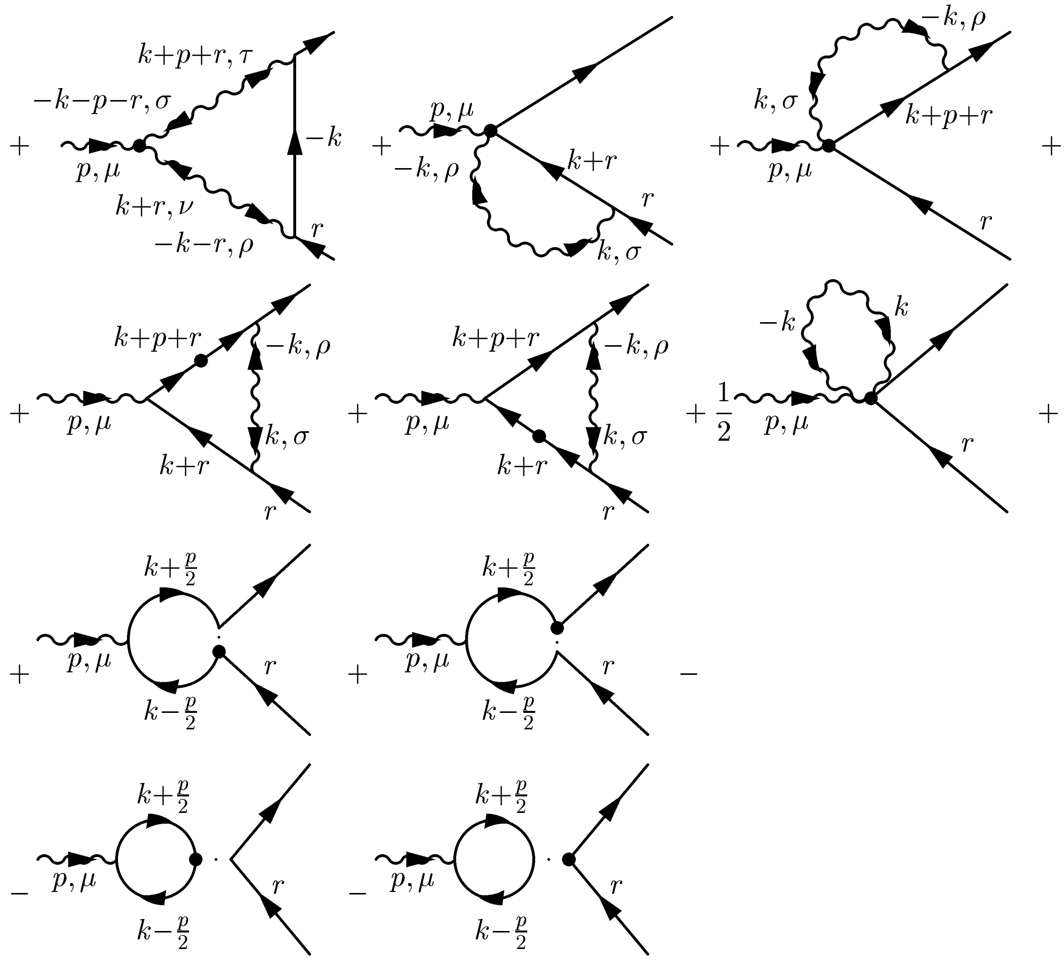
$$\Gamma^{(1,1)}_{\bar\psi\psi;\bar\psi\psi}(p,q;r,s) = \frac{\hbar g^2}{(4\pi)^2\varepsilon}\left(\frac{1}{2}N_\psi - \frac{4}{3}g^2\frac{\partial}{\partial g^2} + \tau\theta^{\alpha\beta}\frac{\partial}{\partial\theta^{\alpha\beta}} + \right. \tag{8.3'}$$
$$\left. + \left(\frac{3}{4}\zeta + \left(-\tau + \frac{4}{3}\right)\kappa_9\right)\frac{\partial}{\partial\kappa_9}\right)\Gamma^{(1,0)}_{\bar\psi\psi;\bar\psi\psi}(p,q;r,s) + \mathcal{O}(1)\,.$$

8.3 Two electrons, one photon

Now we turn to the computation of the electron-photon vertex to first order in θ:

$\Gamma^{(1,1)\mu}_{A\bar\psi\psi}(p,q,r) =$





$$
\begin{aligned}
&= \frac{\hbar g^2}{(4\pi)^2\varepsilon}\Bigg(\frac{1}{2}N_\psi + 0\,N_A + 3m\frac{\partial}{\partial m} + \tau\theta^{\alpha\beta}\frac{\partial}{\partial\theta^{\alpha\beta}} + \\
&\quad + \left(-\frac{13}{48} - \frac{1}{2}\lambda_1 + \frac{1}{6}\lambda_2 + (-\tau - \frac{1}{6})\kappa_1 + \frac{1}{2}\kappa_2 - \frac{4}{3}\kappa_3 - \frac{3}{2}\kappa_4 - \frac{2}{3}\kappa_9\right)\frac{\partial}{\partial\kappa_1} + \\
&\quad + \left(-\frac{1}{12} + \lambda_1 - \frac{1}{3}\lambda_2 + \frac{2}{3}\kappa_1 - \tau\kappa_2 - \frac{2}{3}\kappa_3 + 3\kappa_4 - \frac{1}{3}\kappa_9\right)\frac{\partial}{\partial\kappa_2} + \\
&\quad + \left(\frac{13}{32} + \frac{5}{24}\lambda_2 - \frac{1}{4}\kappa_1 - \frac{1}{4}\kappa_2 - \tau\kappa_3 + \frac{3}{4}\kappa_4\right)\frac{\partial}{\partial\kappa_3} + \\
&\quad + \left(\frac{1}{48} - \frac{1}{6}\kappa_1 + \frac{1}{2}\kappa_2 + \frac{2}{3}\kappa_3 + \left(-\tau - \frac{1}{2}\right)\kappa_4 + \frac{1}{3}\kappa_9\right)\frac{\partial}{\partial\kappa_4} + \\
&\quad + \left(\frac{1}{6} + \frac{2}{3}\kappa_1 - \kappa_2 - \frac{2}{3}\kappa_3 - 3\kappa_4 + (-\tau - 3)\kappa_6 + 2\kappa_7 - \frac{1}{3}\kappa_9\right)\frac{\partial}{\partial\kappa_6} + \\
&\quad + \left(-\frac{1}{2}\kappa_2 - \kappa_3 - \frac{3}{2}\kappa_4 + \frac{1}{2}\kappa_6 + (-\tau - 3)\kappa_7 + \frac{1}{2}\kappa_9\right)\frac{\partial}{\partial\kappa_7} + \\
&\quad + \left(\frac{1}{16} - \frac{1}{2}\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_6 + 2\kappa_7 + (-\tau - 6)\kappa_8 + \frac{1}{2}\kappa_9\right)\frac{\partial}{\partial\kappa_8} + \\
&\quad + \left(\frac{3}{4}\zeta + \left(-\tau + \frac{4}{3}\right)\kappa_9\right)\frac{\partial}{\partial\kappa_9}\Bigg)\left(\Gamma^{(1,0)\mu}_{A\bar\psi\psi}(p,q,r)\right) + \qquad (8.4a)
\end{aligned}
$$

$$+\frac{\hbar g^2}{(4\pi)^2\varepsilon}\mathrm{i}\theta^{\alpha\beta}\left(\left(3+5\lambda_1-\frac{10}{3}\lambda_2\right)mp_\beta g^\mu_\alpha+\left(\frac{3}{2}+\frac{5}{6}\lambda_2\right)mp_\nu\gamma^{\mu\nu}{}_{\alpha\beta}+\right.$$
$$+\left(-2\lambda_1+\frac{2}{3}\lambda_2+\frac{1}{4}\tau\right)\left((2g^\mu_\nu p_\alpha-2g^\mu_\alpha p_\nu)r_\beta-g^\mu_\alpha p_\nu p_\beta\right)\gamma^\nu+$$
$$\left.+\left(\frac{3}{4}\zeta-\frac{1}{2}\lambda_1+\frac{1}{6}\lambda_2\right)(p^2g^{\mu\nu}-p^\mu p^\nu)\gamma_{\nu\alpha\beta}\right)+\mathcal{O}(1)\,. \qquad (8.4b)$$

There are several observations:

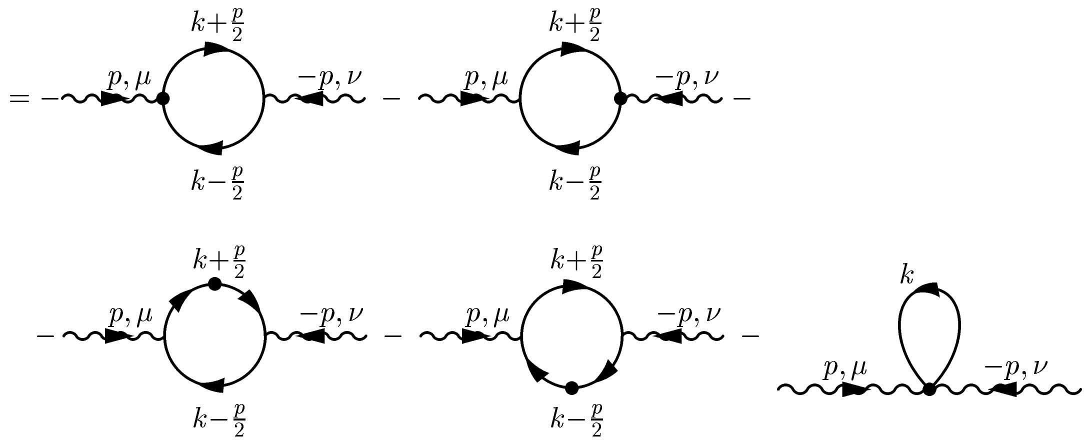
- The renormalizations of $\kappa_4, \kappa_6, \kappa_7, \kappa_8, \kappa_9$ are the same in (8.2), (8.3) and (8.4a), which together with the transversality of the additional terms in (8.4b) verifies the Ward identity (see the next section).
- The field redefinitions parametrized by κ_i always give rise to renormalizations of κ_j, never to a renormalization of the "physical" counterterms required by (8.4b).
- We obtain divergences in (8.4b) which have no counterpart in the original action (5.3), in particular, they cannot be field redefinitions. In the massive case not all of them can be eliminated.
- In the massless case $m = 0$ both remaining divergences in (8.4b) are absent for $6\lambda_1 - 2\lambda_2 = 9\zeta + \mathcal{O}(\hbar)$ and $\tau = 12\zeta + \mathcal{O}(\hbar)$.

The last remark means that for $\zeta = 0$ (mathematical framework) in the massless case no θ-renormalization and no extension terms (6.10) to the bosonic action are required. Thus we have the choice of a serious problem in (8.3) or in (8.4). It is clear that we prefer to make (8.4) as nice as possible.

8.4 No electrons, two photons

It remains to compute the divergent one-loop graphs without external fermion lines to first order in θ. The exciting question is whether these divergences are compatible with $\lambda_1 = \lambda_2 = 0$ for $\tau = 12\zeta = 0$. For the photon two-point function we obtain

$\Gamma^{(1,1)\mu\nu}_{AA}(p,q) =$



$= \mathcal{O}(1)\,.$ (8.5)

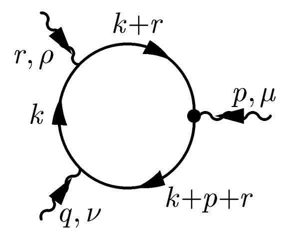
In (8.5) the divergent parts of the first, second and fifth graphs are identically zero and those of the third and fourth graphs are antisymmetric in $(p,\mu) \leftrightarrow (-p,\nu)$ so that their sum is zero. The result of (8.5) was clear from the beginning because there is no gauge-invariant purely bosonic action part to first order in θ.

8.5 No electrons, three photons

Now we compute the photon three-point function to first order in θ:

$\Gamma^{(1,1)\mu\nu\rho}_{AAA}(p,q,r) =$

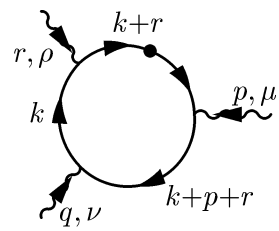
$= \Bigg\{ \Bigg\{ -$



$\Bigg\} + \Big\{ (q,\nu) \leftrightarrow (r,\rho) \Big\} \Bigg\} +$

$+ \Big\{ (p,\mu) \to (q,\nu) \to (r,\rho) \to (p,\mu) \Big\} + \Big\{ (p,\mu) \to (r,\rho) \to (q,\nu) \to (p,\mu) \Big\} +$

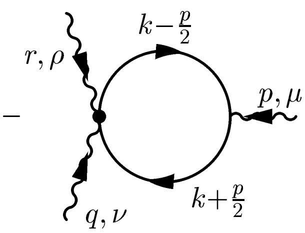
$+ \Bigg\{ \Bigg\{ -$



$\Bigg\} + \Big\{ (q,\nu) \leftrightarrow (r,\rho) \Big\} \Bigg\} +$

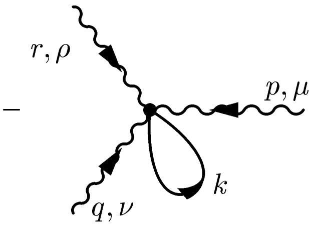
$+ \Big\{ (p,\mu) \to (q,\nu) \to (r,\rho) \to (p,\mu) \Big\} + \Big\{ (p,\mu) \to (r,\rho) \to (q,\nu) \to (p,\mu) \Big\} +$

$+ \Bigg\{ -$



$\Bigg\} + \Big\{ (p,\mu) \leftrightarrow (r,\rho) \Big\} + \Big\{ (p,\mu) \leftrightarrow (q,\nu) \Big\} -$

$-$



$$= \frac{\hbar g^2}{(4\pi)^2\varepsilon} \bigg(0\, N_A - \frac{4}{3} g^2 \frac{\partial}{\partial g^2} + 12\zeta\theta^{\alpha\beta} \frac{\partial}{\partial\theta^{\alpha\beta}} + \Big(-\frac{4}{3}\lambda_1 - 12\zeta(1{+}\lambda_1) \Big) \frac{\partial}{\partial\lambda_1} + \\ + \Big(-\frac{4}{3}\lambda_2 - 12\zeta(1{+}\lambda_2) \Big) \frac{\partial}{\partial\lambda_2} \bigg) \Gamma^{(1,0)\mu\nu\rho}_{AAA;1+2}(p,q,r) + \mathcal{O}(1)\,. \tag{8.6}$$

We confirm indeed that for $\zeta = 0$ the choice $\lambda_1 = \lambda_2 = 0$ in (6.10), which corresponds to the θ-expansion of the unmodified non-commutative Yang-Mills-Dirac action (3.1), is stable at one-loop level. A deeper understanding of this result is missing. It points however to a link between the bosonic action and an effective fermion action as in [31]. Before entering the discussion we have to prove that this nice result (ignoring the κ_9-problem) is not going to change by the remaining divergent Green's functions.

9. Ward identities

The action $\Sigma = \Sigma_{c\ell} + \Sigma_{gf}$ given in (3.1) and (5.4a) is invariant under abelian BRST transformations (5.5). Switching to momentum space, functional derivation with respect to $c(p)$ and restriction to the physical sector $\{B, \bar{c}, c\} = 0$ leads to the following form of the Ward identity:

$$\left[p_\mu \frac{\delta\Gamma}{\delta A_\mu(p)} + \int \frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}(2\pi)^4\delta(p{+}q{+}r)\left(\bar{\psi}(q)\frac{\overrightarrow{\delta}\,\Gamma}{\delta\bar{\psi}(-r)} - \frac{\Gamma\overleftarrow{\delta}}{\delta\psi(-q)}\psi(r)\right)\right]_{\{B,\bar{c},c\}=0} = \\ = 0\,. \tag{9.1}$$

From (9.1) we derive the following identities:

$$0 = \left[p_\mu \frac{\delta^n}{\delta A_{\nu_1}(q_1)\dots\delta A_{\nu_n}(q_n)}\frac{\delta\Gamma}{\delta A_\mu(p)}\right]_{\{A,\bar{\psi},\psi,B,\bar{c},c\}=0}, \tag{9.2a}$$

$$\begin{aligned}0 = \Bigg[&p_\mu \frac{\overrightarrow{\delta}}{\delta\bar{\psi}(q)}\frac{\delta^{n+1}\Gamma}{\delta A_\mu(p)\delta A_{\nu_1}(p_1)\dots\delta A_{\nu_n}(p_n)}\frac{\overleftarrow{\delta}}{\delta\psi(r)} + \\ &+ \frac{\overrightarrow{\delta}}{\delta\bar{\psi}(p{+}q)}\frac{\delta^n\Gamma}{\delta A_{\nu_1}(p_1)\dots\delta A_{\nu_n}(p_n)}\frac{\overleftarrow{\delta}}{\delta\psi(r)} - \\ &- \frac{\overrightarrow{\delta}}{\delta\bar{\psi}(q)}\frac{\delta^n\Gamma}{\delta A_{\nu_1}(p_1)\dots\delta A_{\nu_n}(p_n)}\frac{\overleftarrow{\delta}}{\delta\psi(p{+}r)}\Bigg]_{\{A,\bar{\psi},\psi,B,\bar{c},c\}=0},\end{aligned} \tag{9.2b}$$

which means

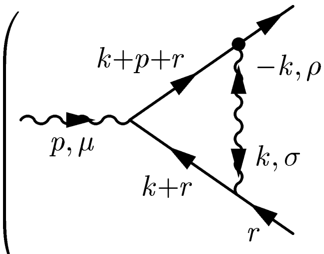
$$p_\mu\Gamma^{(\tau,\ell)\mu\nu_1\dots\nu_n}_{A\dots A}(p,q_1,\dots,q_n) = 0\,, \tag{9.3a}$$

$$p_\mu\Gamma^{(\tau,\ell)\mu}_{A\bar{\psi}\psi}(p,q,r) = \Gamma^{(\tau,\ell)}_{\bar{\psi}\psi}(q,p{+}r) - \Gamma^{(\tau,\ell)}_{\bar{\psi}\psi}(p{+}q,r)\,, \tag{9.3b}$$

$$p_\mu\Gamma^{(\tau,\ell)\mu\nu}_{AA\bar{\psi}\psi}(p,q,r,s) = \Gamma^{(\tau,\ell)\nu}_{A\bar{\psi}\psi}(q,r,p{+}s) - \Gamma^{(\tau,\ell)\nu}_{A\bar{\psi}\psi}(q,p{+}r,s)\,, \tag{9.3c}$$

$$p_\mu\Gamma^{(\tau,\ell)\mu\nu\rho}_{AAA\bar{\psi}\psi}(p,q,r,s,t) = \Gamma^{(\tau,\ell)\nu\rho}_{AA\bar{\psi}\psi}(q,r,s,p{+}t) - \Gamma^{(\tau,\ell)\nu\rho}_{AA\bar{\psi}\psi}(q,r,p{+}s,t)\,. \tag{9.3d}$$

On tree-level $(\tau, 0)$ the identities (9.3) are easy to verify. Let us first investigate (9.3b) for $(\tau, \ell) = (1, 1)$. We perform the manipulations directly on the integrals encoded in the Feynman graphs. Denoting by the subscript $1/\varepsilon$ the divergent part in ε in analytic regularization, let us consider

p_μ 

$$\left(\vphantom{\int} \right)_{1/\varepsilon} =$$

$$
\begin{aligned}
&= \bigg(\int \frac{d^4k}{(2\pi)^4}\, \Gamma^{(1,0)\rho}_{A\bar\psi\psi}(-k,-p-r,k+p+r)\, \times \\
&\quad \times \frac{-\gamma^\alpha (k+p+r)_\alpha - m}{(k+p+r)^2 - m^2 + \mathrm{i}\epsilon} p_\mu \gamma^\mu\, \frac{-\gamma^\beta (k+r)_\beta - m}{(k+r)^2 - m^2 + \mathrm{i}\epsilon} \gamma^\sigma \Delta^{AA}_{(0,0)\sigma\rho}(k,-k) \bigg)_{1/\varepsilon} \\
&= \bigg(\int \frac{d^4k}{(2\pi)^4}\, \Gamma^{(1,0)\rho}_{A\bar\psi\psi}(-k,-p-r,k+p+r) \times \\
&\quad \times \bigg(\frac{(-\gamma^\alpha (k+p+r)_\alpha - m)(\gamma^\mu (k+p+r)_\mu - m)}{(k+p+r)^2 - m^2 + \mathrm{i}\epsilon}\, \frac{-\gamma^\beta (k+r)_\beta - m}{(k+r)^2 - m^2 + \mathrm{i}\epsilon} - \\
&\qquad - \frac{-\gamma^\alpha (k+p+r)_\alpha - m}{(k+p+r)^2 - m^2 + \mathrm{i}\epsilon}\, \frac{(\gamma^\mu (k+r)_\mu - m)(-\gamma^\beta (k+r)_\beta - m)}{(k+r)^2 - m^2 + \mathrm{i}\epsilon} \bigg) \gamma^\sigma \Delta^{AA}_{(0,0)\sigma\rho}(k,-k) \bigg)_{1/\varepsilon} \\
&= - \bigg(\int \frac{d^4k}{(2\pi)^4}\, \Gamma^{(1,0)\rho}_{A\bar\psi\psi}(-k,-p-r,k+p+r) \frac{-\gamma^\beta (k+r)_\beta - m}{(k+r)^2 - m^2 + \mathrm{i}\epsilon} \gamma^\sigma \Delta^{AA}_{(0,0)\sigma\rho}(k,-k) \bigg)_{1/\varepsilon} + \\
&\quad + \bigg(\int \frac{d^4k}{(2\pi)^4}\, \Gamma^{(1,0)\rho}_{A\bar\psi\psi}(-k,-p-r,k+p+r) \frac{-\gamma^\alpha (k+p+r)_\alpha - m}{(k+p+r)^2 - m^2 + \mathrm{i}\epsilon} \gamma^\sigma \Delta^{AA}_{(0,0)\sigma\rho}(k,-k) \bigg)_{1/\varepsilon} .
\end{aligned}
$$

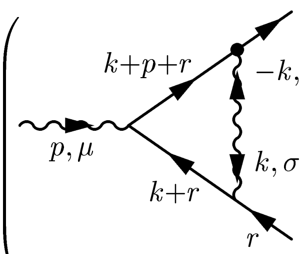
We have used the following property of analytic regularization, see appendix A:

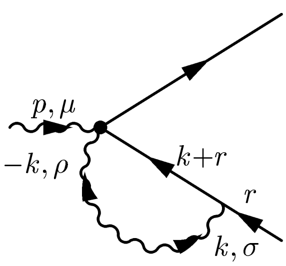
$$
\begin{aligned}
&\bigg(\int \frac{d^4k}{(2\pi)^4}\, \frac{(g^{\mu\nu}(k+p)_\mu (k+p)_\nu - m^2) k_{\rho_1} \dots k_{\rho_n}}{((k+p)^2 - m^2 + \mathrm{i}\epsilon) \prod_{i=1}^s ((k+p_i)^2 - m_i^2 + \mathrm{i}\epsilon)} \bigg)_{1/\varepsilon} = \\
&\qquad = \bigg(\int \frac{d^4k}{(2\pi)^4}\, \frac{k_{\rho_1} \dots k_{\rho_n}}{\prod_{i=1}^s ((k+p_i)^2 - m_i^2 + \mathrm{i}\epsilon)} \bigg)_{1/\varepsilon} .
\end{aligned}
\tag{9.4}
$$

Using the tree-level Ward identity (9.3b),

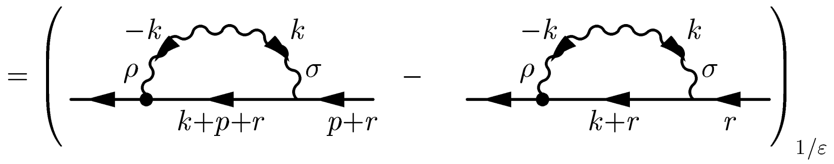
$$\Gamma^{(1,0)\rho}_{A\bar\psi\psi}(-k,-p-r,k+p+r) = p_\mu \Gamma^{(1,0)\mu\rho}_{AA\bar\psi\psi}(p,-k,-p-r,k+r) + \Gamma^{(1,0)\rho}_{A\bar\psi\psi}(-k,-r,k+r)\,,$$

we conclude

p_μ 

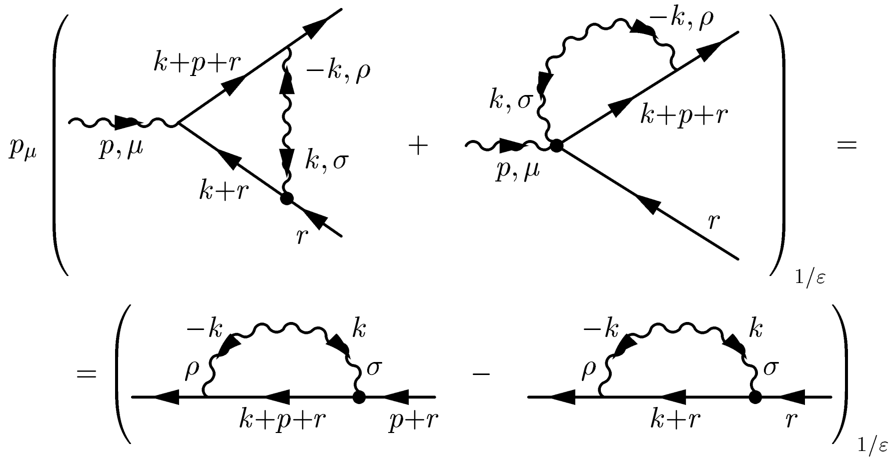
$+$ 

$$\left(\vphantom{\int} \right)_{1/\varepsilon} =$$

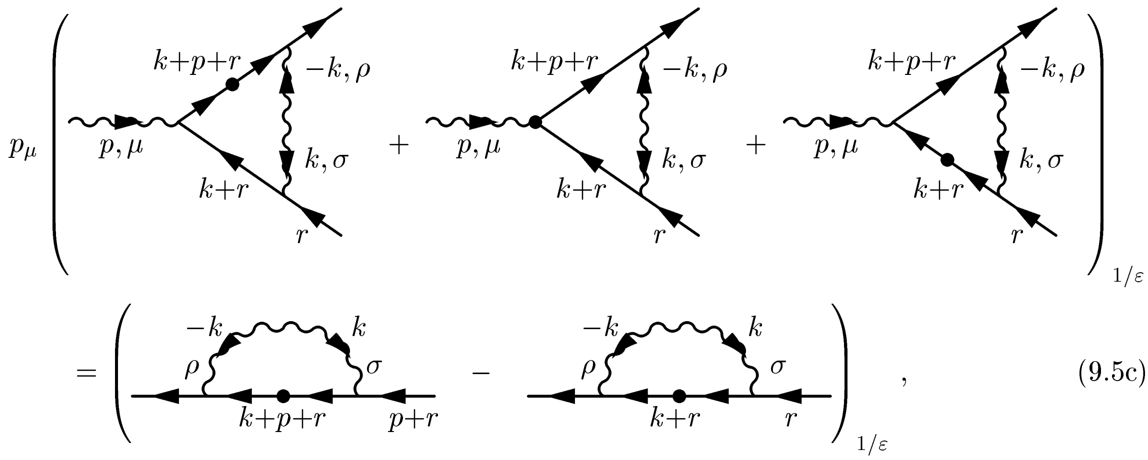


. (9.5a)

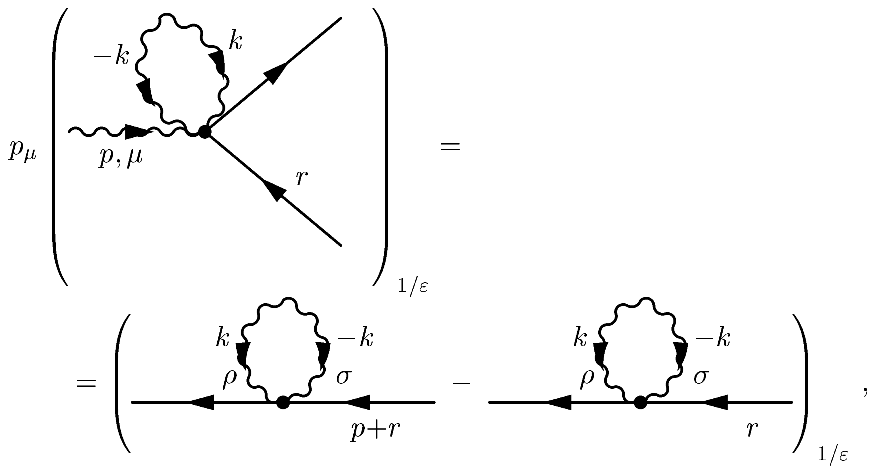
In the same way one proves



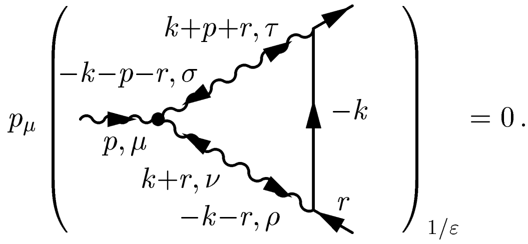
, (9.5b)



, (9.5c)



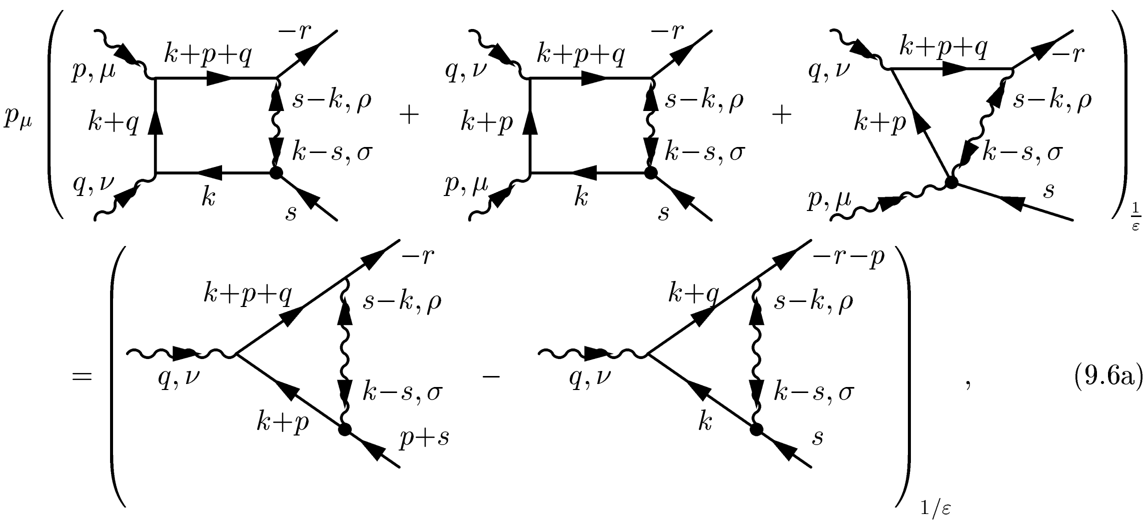
, (9.5d)



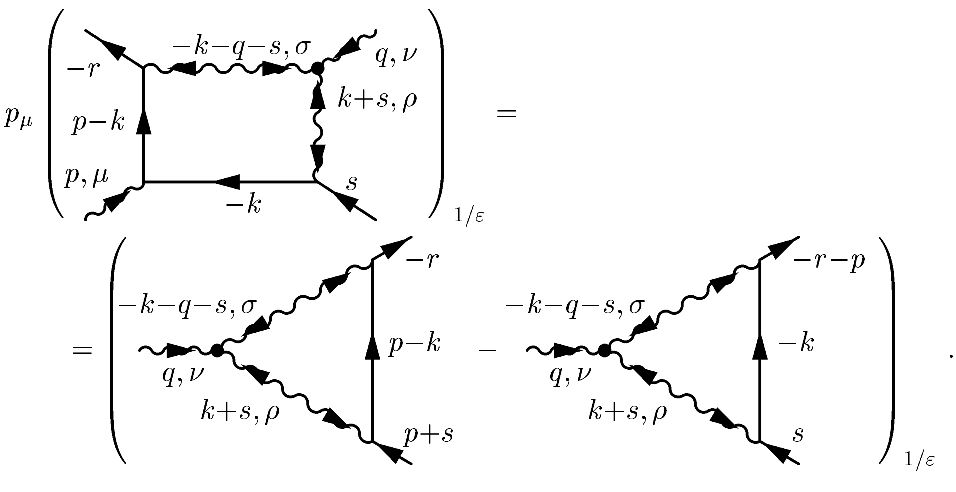
(9.5e)

The four graphs in (8.4) which involve the four-fermion vertex turn out to be transversal (contraction with p_μ yields zero). That zero can formally be written as the difference of the corresponding graphs in (8.2) containing the four-fermion vertex, with external momenta $p+r$ and r, respectively, because the singular part of these graphs is independent of the external momentum. Thus, (9.5) proves the Ward identity (9.3b) for $(\tau,\ell)=(1,1)$. We stress that this proof only uses the possibility of a naive factorization (9.4) of common terms in numerator and denominator of the integrand, valid for analytic regularization. It is not necessary to evaluate the divergent integrals.

In the same way we can prove the Ward identities (9.3c) and (9.3d) without computing the (already very complicated) divergent integrals. For instance, we have



, (9.6a)

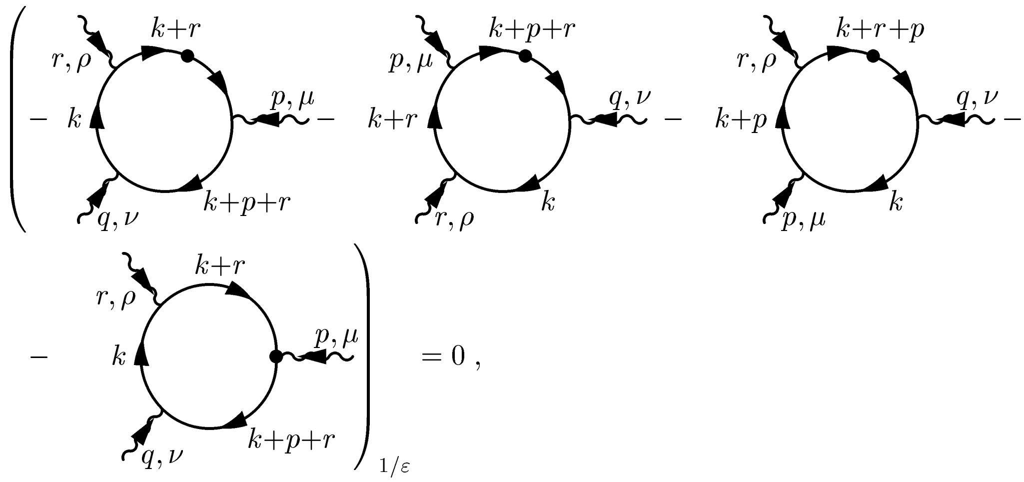


. (9.6b)

Thus, the evaluation of the divergent part of $\Gamma^{(1,1)\mu\nu}_{AA\bar\psi\psi}(p,q,r,s)$ and $\Gamma^{(1,1)\mu\nu\rho}_{AAA\bar\psi\psi}(p,q,r,s,t)$ is compatible with the $\hbar$-renormalizations of κ_i and the additional counterterms required by (8.4b).

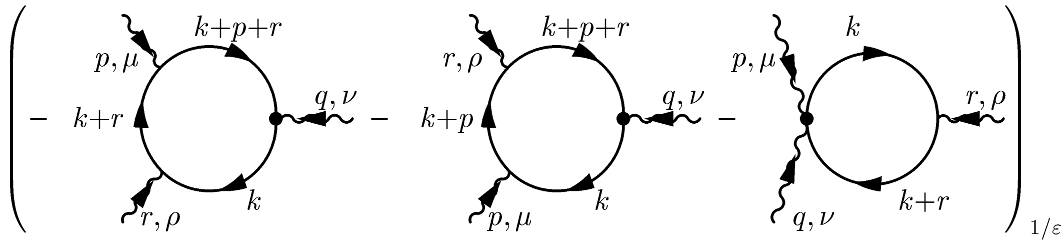
Let us finally show (9.3a) for $n=2$, the generalization to higher n being obvious. We have

p_μ



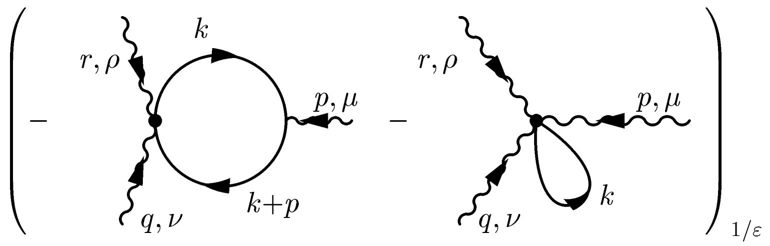
$= 0\,,$ (9.7a)

p_μ



$= 0\,,$ (9.7b)

p_μ



$= 0\,.$ (9.7c)

By exchange $(q,\nu)\leftrightarrow(r,\rho)$ in (9.7a) and (9.7b), all graphs contributing to (8.6) are obtained, which proves (9.3a) for $n=2$.

Proceeding analogously one proves (9.3a) for $n\in\{3,4,5\}$. But this means that the coefficient of $1/\varepsilon$ in analytic regularization of the 1-loop Green's functions $\Gamma^{(1,1)\mu_0\dots\mu_n}_{A\dots A}(p_0,\dots,p_n)$ is the Fourier-transformed of a gauge-invariant local field polynomial

$$\sum\int d^4x\prod_{i=1}^{n+1}d^4x_i\;\theta\,(\partial\dots\partial F)(x_1)\dots(\partial\dots\partial F)(x_{n+1})\prod_{j=1}^{n+1}\delta^4(x-x_j)\,,\qquad(9.8)$$

for an appropriate contraction of Lorentz indices. On the other hand, the integral has to be of power-counting dimension zero (see footnote 13), which cannot be achieved for $n>2$.

This means

$$\Gamma^{(1,1)\mu_0\ldots\mu_n}_{A\ldots A}(p_0,\ldots,p_n) = \mathcal{O}(1)\,, \qquad \text{for} \;\; n > 2\,. \tag{9.9}$$

Individual graphs contributing to $\Gamma^{(1,1)\mu_0\ldots\mu_n}_{A\ldots A}(p_0,\ldots,p_n)$ for $n \in \{3,4,5\}$ will be divergent, of course.

10. Discussion

We have computed or derived all divergent one-loop corrections to Green's functions of θ-expanded non-commutative QED, up to first order in θ. Let us summarize the results:

1) Taking the θ-expansion serious, the model is obviously *not renormalizable*. This is the death of all attempts to avoid considering the full non-commutative quantum field theory by Seiberg-Witten expansion.[15]

   The problem is first of all due to the divergence of the fermion four-point function. From a mathematically appealing point of view, fermions are elements of an inner product space. Therefore, any *local* field monomial can never contain more than two fermion fields, which means that divergences in graphs with more than two external fermions cannot be renormalized. At order T in θ, graphs with N_F external fermion lines are divergent for $3N_F \leq 8{+}4T$. There is no reason why this infinite number of divergences could cancel for a quantum field theoretical model with a finite number of fields. Anyway, the inclusion of (on non-commutative level non local) fermion number-changing field redefinitions does not yield a renormalizable quantum field theory either.

2) Let us "solve" the above problem in graphs with more than two external electrons by ignoring it (to be made precise below). Then θ-expanded non-commutative QED is not renormalizable if the electrons are massive, with the mass term appearing explicitly in the non-commutative Dirac action. This does not exclude a fermion mass coming from a Higgs mechanism. It would therefore be important to study an abelian Higgs model.

3) Let us therefore consider θ-expanded *massless* non-commutative QED with the divergence in graphs with $n > 2$ external electrons being ignored. We have proved that in this case our model (3.1) is multiplicatively one-loop renormalizable — including field redefinitions — up to first order in θ. This $\hbar$-dependence of the parameters of the model is given by (7.6) and

$$\begin{aligned}
\kappa_1 &= \kappa_{1,0} - \frac{\hbar g_0^2}{(4\pi)^2\varepsilon}\left(-\frac{13}{48} - \frac{1}{6}\kappa_{1,0} + \frac{1}{2}\kappa_{2,0} - \frac{4}{3}\kappa_{3,0} - \frac{3}{2}\kappa_{4,0}\right) + \mathcal{O}(\hbar^2)\,,\\
\kappa_2 &= \kappa_{2,0} - \frac{\hbar g_0^2}{(4\pi)^2\varepsilon}\left(-\frac{1}{12} + \frac{2}{3}\kappa_{1,0} - \frac{2}{3}\kappa_{3,0} + 3\kappa_{4,0}\right) + \mathcal{O}(\hbar^2)\,,\\
\kappa_3 &= \kappa_{3,0} - \frac{\hbar g_0^2}{(4\pi)^2\varepsilon}\left(\frac{13}{32} - \frac{1}{4}\kappa_{1,0} - \frac{1}{4}\kappa_{2,0} + \frac{3}{4}\kappa_{4,0}\right) + \mathcal{O}(\hbar^2)\,,
\end{aligned}$$

[15] It could still be meaningful to consider θ-expansions of non-commutative field theories as effective actions.

$$
\begin{aligned}
\kappa_4 &= \kappa_{4,0} - \frac{\hbar g_0^2}{(4\pi)^2\varepsilon}\left(\frac{1}{48} - \frac{1}{6}\kappa_{1,0} + \frac{1}{2}\kappa_{2,0} + \frac{2}{3}\kappa_{3,0} - \frac{1}{2}\kappa_{4,0}\right) + \mathcal{O}(\hbar^2)\,, \\
\kappa_6 &= \kappa_{6,0} - \frac{\hbar g_0^2}{(4\pi)^2\varepsilon}\left(\frac{1}{6} + \frac{2}{3}\kappa_{1,0} - \kappa_{2,0} - \frac{2}{3}\kappa_{3,0} - 3\kappa_{4,0} - 3\kappa_{6,0} + 2\kappa_{7,0}\right) + \mathcal{O}(\hbar^2)\,, \\
\kappa_7 &= \kappa_{7,0} - \frac{\hbar g_0^2}{(4\pi)^2\varepsilon}\left(-\frac{1}{2}\kappa_{2,0} - \kappa_{3,0} - \frac{3}{2}\kappa_{4,0} + \frac{1}{2}\kappa_{6,0} - 3\kappa_{7,0}\right) + \mathcal{O}(\hbar^2)\,, \\
\kappa_8 &= \kappa_{8,0} - \frac{\hbar g_0^2}{(4\pi)^2\varepsilon}\left(\frac{1}{16} - \frac{1}{2}\kappa_{1,0} + \kappa_{2,0} + \kappa_{3,0} + \kappa_{4,0} + \kappa_{6,0} + 2\kappa_{7,0} - 6\kappa_{8,0}\right) + \\
&\quad + \mathcal{O}(\hbar^2)\,. \qquad (10.1)
\end{aligned}
$$

But there is no reason why Green's functions with not more than two external electrons are renormalizable up to the considered order. There could be four additional divergences which are allowed by gauge symmetry (Ward identity) and Lorentz symmetry — two in the pure photon sector, one corresponding to a renormalization of θ and one in the photon-electron sector. All of these four divergences are absent! This cannot be accidental!

Our task is now to start developing a renormalization scheme for non-commutative gauge theories implementing these results. The divergences in graphs with more than two external electrons discussed in 1) tell us that one *must not expand* the non-commutative field theory in θ. Then only graphs with

$$
\hat{N}_B + \frac{3}{2}\hat{N}_F \leq 4 \qquad (10.2)
$$

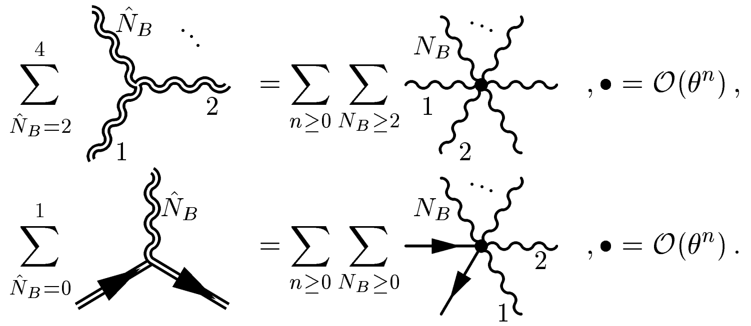
are divergent, where $\hat{N}_B$ and $\hat{N}_F$ are the numbers of (θ-unexpanded) external gauge bosons and fermions, respectively.[16] In this way we solve 1) automatically. One might object immediately that now the famous UV/IR mixing destroys renormalizability. However, our results 3) tell us to be more careful, although the problem seems to persist.

Actually the UV/IR mixing [15] was due to the distinction of the Feynman graphs into *planar* and *non-planar* ones. The custom was to subtract the planar graphs as usual by *local counterterms* and to keep the non-planar graphs untouched, because non-planar graphs (seem to) correspond to non-local counterterms [18]. The trouble came when inserting the non-planar (at first sight finite) graphs as subgraphs into a bigger graph, which then turned out to be divergent and non local. It is at this point where our results propose to modify the subtraction scheme. We have seen that there is a way to *subtract the non-planar graphs* at least partially *without destroying the symmetries* — using the Seiberg-Witten map in a crucial way.

It is convenient to represent this idea graphically. Let us draw Feynman rules for the θ-unexpanded model as double lines. The Seiberg-Witten differential equation (including all possible field redefinitions) expands the non-commutative Yang-Mills-Dirac action in

[16] In other words, the divergence of the fermion four-point function should be regarded as an artifact of the θ-expansion. Note, however, that similar divergences will appear in the θ-unexpanded theory if θ-dependent field monomials such as $\int \theta^{\alpha\beta}\hat{F}_{\alpha\beta}\star\hat{F}_{\mu\nu}\star\hat{F}^{\mu\nu}$ are included. These terms are perfectly compatible with power-counting dimension and gauge and Lorentz symmetries!

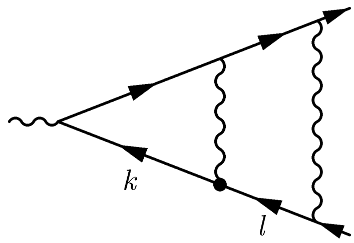
the following way:



(10.3)

A superficially divergent Green's function $\hat{\Gamma}$ of the θ-unexpanded theory will then via Seiberg-Witten map be expressed in terms of Green's functions Γ of the θ-expanded theory. Renormalization has to proceed order by order in $\hbar$ (which is the number of loops). In order $\hbar^1$ we have proved that massless QED is renormalizable up to first order of θ, because graphs with more than two external fermions are not expanded. Let us assume that one-loop renormalizability can be extended to any order of θ. Under this assumption all divergences would have been removed for a certain $\hbar^1$-renormalization of the initial non-commutative Yang-Mills-Dirac action.

Starting with order $\hbar^2$ there is however a new problem to solve. Let us consider the graph



. (10.4)

This graph contains an overlapping divergence in θ-expanded non-commutative QED, because the fermion four-point function is divergent. In a (renormalizable) quantum field theory on commutative space-time local counterterms are obtained only if *all* subdivergences are treated according to the forest formula. Otherwise there are divergent terms involving logarithms of external momenta and masses. For the graph (10.4), however, we are not allowed to treat the divergent l-integration as a subdivergence in the forest formula. One has therefore to show that first integrating over k, subtracting then the divergent part, and finally integrating over l does not yield divergent terms which contain logarithms of external momenta — at least on the level of Green's functions. It seems unlikely that this can work, but one cannot exclude the possibility of symmetries for the θ-deformed action which do the job. We had also expected four additional divergences at order $\hbar^1\theta^1$ which eventually were absent. Thus, one has to perform the two-loop computation in order to be sure.

11. Symmetries

We have mentioned several times possible (additional) symmetries of gauge theories on θ-deformed space-time. There are clear hints now that such symmetries exist. Otherwise the absence of the four divergences which are not reached by field redefinitions cannot be explained.[17] A general idea about these symmetries can be obtained from the mathematical foundation [8] of non-commutative geometry. The non-commutative Dirac action can be written identically as

$$\langle\hat{\psi},(\mathrm{i}\not{\partial}+\hat{\not{A}})\star\hat{\psi}\rangle_{\mathcal{H}}=\langle U\star\hat{\psi},(\mathrm{i}\not{\partial}+U\star\hat{\not{A}}\star U^*+U\star[\mathrm{i}\not{\partial},U^*])\star(U\star\hat{\psi})\rangle_{\mathcal{H}} \tag{11.1}$$

for $U\in\mathcal{M}$, $U\star U^*=U^*\star U=1$ (see (2.3)). One is tempted to regard

$$\hat{\not{A}}\mapsto\hat{\not{A}}_U:=U\star\hat{\not{A}}\star U^*+U\star[\mathrm{i}\not{\partial},U^*]\,,\qquad\hat{\psi}\mapsto\hat{\psi}_U:=U\star\hat{\psi}\,, \tag{11.2}$$

as a gauge transformation. However, the space of all U is very big, for instance, Lorentz transformations can be implemented via U [28]. In fact all automorphisms of the algebra $\mathcal{A}$ are inner [33]. Thus, U is the candidate for additional symmetries. However, one has to make sure that the non-commutative Yang-Mills action is invariant under (11.2). Since we look for an action invariant under all automorphisms, an action like $\int\hat{\not{F}}\star\hat{\not{F}}$ which does not contain gravity (probably in a different shape) is certainly not the right choice. There is — at least formally — a natural candidate for a Yang-Mills action invariant under (11.2), the spectral action [34]

$$S_\Lambda(\hat{\not{A}})=\mathrm{trace}\ f\left(\frac{\not{D}_A^2}{\Lambda^2}\right),\qquad\not{D}_A=\mathrm{i}\not{\partial}+\hat{\not{A}}\,, \tag{11.3}$$

where f is an appropriate cut-off with $f(0)=1$ and $f(x)=0$ for $x\geq1$. This is nothing but the weighted sum of the eigenvalues of $\not{D}_A^2$ smaller than Λ^2. The problem is that we are in a non-compact case so that the eigenvalues of $\not{D}_A^2$ are continuous. Recently there has been a lot of progress on non-compact spectral geometries [28].

Our computations provide an indirect support for the spectral action. What we have computed in section 8.5 is (together with (7.4)) the divergent part of the effective action for fermions coupled to an external photon field, the coupling involving θ. In the commutative case one recovers the Maxwell action $-\frac{1}{4g^2}\int F_{\mu\nu}F^{\mu\nu}$ as the coefficient of that (logarithmic) divergence. This result was rigorously proved by Langmann [31] in the language of regularized traces for pseudodifferential operators, i.e. similar techniques as those which are used to evaluate the spectral action. There is some rumour that Langmann's work and the spectral action are equivalent when restricted to the Yang-Mills part. Whereas in the commutative case gauge and Lorentz symmetries do not permit a different coefficient of the divergence than the Maxwell action, we could in the θ-expanded non-commutative case obtain in principle the additional terms (6.10). But this does not happen. We expect therefore that the spectral action forbids these additional terms trilinear in the non-commutative field strength. Thus, making sense of (11.3), computing the spectral action and identifying the additional local symmetries of the spectral action is one of the most important next steps for the renormalization of non-commutative gauge theories.

[17] These symmetries could also be reductions of couplings [32] only.

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Acknowledgments

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A. Analytic regularization

The one-loop integrals to compute are of the form

$$I^{(s+1)}_{\mu_1\dots\mu_n}(p_0,p_1,\dots,p_s;m_0,m_1,\dots,m_s) := \lim_{\epsilon\to 0}\int \frac{d^4k}{(2\pi)^4}\,\frac{\prod_{l=1}^n k_{\mu_l}}{\prod_{i=0}^s[(k+p_i)^2-m_i^2+\mathrm{i}\epsilon]}\,. \tag{A.1}$$

Using Zimmermann's ϵ-trick we replace a propagator

$$\begin{aligned}\frac{1}{k^2-m^2+\mathrm{i}\epsilon} &\equiv \frac{1}{k_0^2-(\vec{k}^{\,2}+m^2)+\mathrm{i}\epsilon}\\ &\longmapsto \frac{1}{k_0^2-(1-\mathrm{i}\epsilon)(\vec{k}^{\,2}+m^2)} = \frac{(\epsilon'-\mathrm{i})}{(\epsilon'-\mathrm{i})k_0^2+(\epsilon-\epsilon'+\mathrm{i}+\mathrm{i}\epsilon\epsilon')(\vec{k}^{\,2}+m^2)}\,.\end{aligned} \tag{A.2}$$

For $0<\epsilon'<\epsilon$ the denominator of the last expression has a positive real part so that standard euclidean integrations techniques using Schwinger and Feynman parameters can be applied. Analytic regularization [25] consists in the following replacement of the denominator of the integrand:

$$\begin{aligned}&\frac{(\epsilon'-\mathrm{i})^{s+1}}{\big\{(\epsilon'-\mathrm{i})[k_0^2+2k_0q_0+M_0^2]+(\epsilon-\epsilon'+\mathrm{i}+\mathrm{i}\epsilon\epsilon')[\vec{k}^2+2\vec{k}\vec{q}+\vec{M}^2]\big\}^{s+1}}\longmapsto\\ &\longmapsto \frac{\mu^{2\varepsilon}(\epsilon'-\mathrm{i})^{s+1+\varepsilon}}{\big\{(\epsilon'-\mathrm{i})[k_0^2+2k_0q_0+M_0^2]+(\epsilon-\epsilon'+\mathrm{i}+\mathrm{i}\epsilon\epsilon')[\vec{k}^2+2\vec{k}\vec{q}+\vec{M}^2]\big\}^{s+1+\varepsilon}}\,.\end{aligned} \tag{A.3}$$

Then the integration over the loop momentum k and over the Schwinger parameter can be performed, with the following result (which now depends additionally on the renormalization scale μ and on ε):

$$\begin{aligned}&I^{(s+1)}_{\mu_1\dots\mu_n}(p_0,p_1,\dots,p_s;m_0,m_1,\dots,m_s;\mu,\varepsilon) =\\ &= \lim_{\epsilon\to0,\ \epsilon'<\epsilon}\sum_{l=0}^{[n/2]}\frac{(-1)^n\Gamma(s-l-1+\varepsilon)\Gamma(s+1)\mu^{2\varepsilon}(\epsilon'-\mathrm{i})^{s+\frac12-l+\varepsilon}}{2^l(4\pi)^2\Gamma(s+1+\varepsilon)(\epsilon-\epsilon'+\mathrm{i}+\mathrm{i}\epsilon\epsilon')^{3/2}}\times\\ &\quad\times\int d^sx\,T^{n-2l}_{\mu_1\dots\mu_n}\Big(p_0+\sum_{i=1}^s x_i(p_i-p_0)\,,\epsilon\Big)\times\\ &\quad\times\Big\{(\epsilon'-\mathrm{i})\sum_{i,j=1}^s x_i(\delta_{ij}-x_j)(p_{i0}-p_{00})(p_{j0}-p_{00})+(\epsilon-\epsilon'+\mathrm{i}+\mathrm{i}\epsilon\epsilon')\times\\ &\qquad\times\Big(M_s^2(x)+\sum_{i,j=1}^s x_i(\delta_{ij}-x_j)(\vec{p}_i-\vec{p}_0)(\vec{p}_j-\vec{p}_0)\Big)\Big\}^{l+1-s-\varepsilon},\end{aligned} \tag{A.4}$$

where

$$\int d^s x = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ldots \int_0^{1-\sum_{j=1}^{s-1} x_j} dx_s\,, \quad M_s^2(x) = m_0^2 + \sum_{i=1}^{s} x_i (m_i^2 - m_0^2)\,. \tag{A.5}$$

Here the completely symmetric tensors $T^k_{\mu_1\ldots\mu_n}(p)$ are inductively defined by $T^0(p) := 1$ and

$$T^0_{\mu_1\ldots\mu_n}(p,\epsilon) := \sum_{j=2}^{n} g_{\mu_1\mu_j}(\epsilon)\, T^0_{\mu_2\ldots\mu_{j-1}\mu_{j+1}\ldots\mu_n}(p,\epsilon)\,, \tag{A.6a}$$

$$T^k_{\mu_1\ldots\mu_n}(p,\epsilon) := \frac{1}{k}\sum_{j=1}^{n} p_{\mu_j} T^{k-1}_{\mu_1\ldots\mu_{j-1}\mu_{j+1}\ldots\mu_n}(p,\epsilon)\,, \qquad k\in\{n, n-2,\ldots\}, \qquad k \geq 1\,, \tag{A.6b}$$

$$g_{\mu\nu}(\epsilon) = \mathrm{diag}\left(1, -\frac{1}{1-\mathrm{i}\epsilon}, -\frac{1}{1-\mathrm{i}\epsilon}, -\frac{1}{1-\mathrm{i}\epsilon}\right). \tag{A.6c}$$

In particular, $T^1_\mu(p,\epsilon) = p_\mu$ and $T^0_{\mu\nu}(p) = g_{\mu\nu}(\epsilon)$. We define $T^k_{\mu_1\ldots\mu_n}(p) = 0$ for $k < 0$ and $k > n$.

The leading terms in ε of (A.4) are then given by

$$\begin{aligned} &I^{(s+1)}_{\mu_1\ldots\mu_n}(p_0, p_1, \ldots, p_s; m_0, m_1, \ldots, m_s; \mu, \varepsilon) = \\ &\quad = \frac{\mathrm{i}}{(4\pi)^2\varepsilon} \sum_{\delta=0}^{[n/2]-s+1} \frac{(-1)^{n-\delta}}{2^{\delta+s-1}\delta!} \int d^s x\; T^{n-2\delta-2s+2}_{\mu_1\ldots\mu_n}\left(p_0 + \sum_{i=1}^{s} x_i (p_i - p_0)\right) \times \\ &\qquad \times \left\{\sum_{i,j=1}^{s} x_i(\delta_{ij} - x_j)(p_i - p_0)(p_j - p_0) - M_s^2(x)\right\}^\delta + \mathcal{O}(1)\,, \end{aligned} \tag{A.7}$$

$$\begin{aligned} &\mu^2 \frac{d}{d\mu^2}\left(I^{(s+1)}_{\mu_1\ldots\mu_n}(p_0, p_1, \ldots, p_s; m_0, m_1, \ldots, m_s; \mu, \varepsilon)\right) = \\ &\quad = \frac{\mathrm{i}}{(4\pi)^2} \sum_{\delta=0}^{[n/2]-s+1} \frac{(-1)^{n-\delta}}{2^{\delta+s-1}\delta!} \int d^s x T^{n-2\delta-2s+2}_{\mu_1\ldots\mu_n}\left(p_0 + \sum_{i=1}^{s} x_i (p_i - p_0)\right) \times \\ &\qquad \times \left\{\sum_{i,j=1}^{s} x_i(\delta_{ij} - x_j)(p_i - p_0)(p_j - p_0) - M_s^2(x)\right\}^\delta + \mathcal{O}(\varepsilon)\,. \end{aligned} \tag{A.8}$$

We need two important properties of (A.7) and (A.8): (A) Invariance under the shift of the integration momentum and (B) naive factorization of common terms in numerator and denominator.

(A) To formulate shift invariance, let $\mathcal{P}^l_n\{\mu_1\ldots\mu_n\}$ be any subset of l elements of $\{\mu_1,\ldots,\mu_n\}$, preserving the order. Let $\overline{\mathcal{P}^{n-l}_n}\{\mu_1\ldots\mu_n\} = \{\mu_1,\ldots,\mu_n\} \setminus \mathcal{P}^l_n\{\mu_1\ldots\mu_n\}$ be its complement. The empty set and the total set are regarded as subsets. If $\mathcal{P}^l_n\{\mu_1\ldots\mu_n\} = \{\nu_1,\ldots\nu_l\}$ let $q_{\mathcal{P}^l_n\{\mu_1\ldots\mu_n\}} = q_{\nu_1}\cdots q_{\nu_l}$ for $l > 0$ and $q_{\mathcal{P}^0_n\{\mu_1\ldots\mu_n\}} = 1$. According to (A.1), shift invariance means

$$\begin{aligned} &I^{(s+1)}_{\mu_1\ldots\mu_n}(p_0, p_1, \ldots, p_s; m_0, m_1, \ldots, m_s) = \\ &\quad = \sum_{\mathcal{P}^l_n} q_{\mathcal{P}^l_n\{\mu_1\ldots\mu_n\}} I^{(s+1)}_{\overline{\mathcal{P}^{n-l}_n}\{\mu_1\ldots\mu_n\}}(p_0+q, p_1+q, \ldots, p_s+q; m_0, m_1, \ldots, m_s)\,. \end{aligned}$$

For the leading terms in (A.7) and (A.8) this amounts to verify

$$T^{\alpha}_{\mu_1\ldots\mu_n}(p) = \sum_{\mathcal{P}^l_n,\,0\le l\le\alpha} (-1)^l\, q_{\mathcal{P}^l_n\{\mu_1\ldots\mu_n\}}\, T^{\alpha-l}_{\mathcal{P}^{n-l}_n\{\mu_1\ldots\mu_n\}}(p{+}q)\,. \tag{A.9}$$

Equation (A.9) is obvious for $\alpha\in\{0,1\}$ and any n and follows from (A.6b) by induction in α. Assuming it holds for $\alpha{-}1$ and $n{-}1$ we have

$$\begin{aligned}
T^{\alpha}_{\mu_1\ldots\mu_n}(p) &= \frac{1}{\alpha}\sum_{j=1}^{n}(p_{\mu_j}{+}q_{\mu_j})\sum_{\mathcal{P}^l_{n-1}}(-1)^l q_{\mathcal{P}^l_{n-1}\{\mu_1\ldots\mu_{j-1}\mu_j\ldots\mu_n\}}T^{\alpha-1-l}_{\mathcal{P}^{n-1-l}_{n-1}\{\mu_1\ldots\mu_{j-1}\mu_j\ldots\mu_n\}}(p{+}q) - \\
&\quad - \frac{1}{\alpha}\sum_{j=1}^{n} q_{\mu_j}\sum_{\mathcal{P}^l_{n-1}}(-1)^l q_{\mathcal{P}^l_{n-1}\{\mu_1\ldots\mu_{j-1}\mu_j\ldots\mu_n\}}T^{\alpha-1-l}_{\mathcal{P}^{n-1-l}_{n-1}\{\mu_1\ldots\mu_{j-1}\mu_j\ldots\mu_n\}}(p{+}q) \\
&= \sum_{\mathcal{P}^l_n,\,l<n}\frac{\alpha-l}{\alpha}(-1)^l q_{\mathcal{P}^l_n\{\mu_1\ldots\mu_n\}}T^{\alpha-l}_{\mathcal{P}^{n-l}_n\{\mu_1\ldots\mu_n\}}(p{+}q) + \\
&\quad + \sum_{\mathcal{P}^{l+1}_n,\,l<n}\frac{l+1}{\alpha}(-1)^{l+1} q_{\mathcal{P}^{l+1}_n\{\mu_1\ldots\mu_n\}}T^{\alpha-l-1}_{\mathcal{P}^{n-l-1}_n\{\mu_1\ldots\mu_n\}}(p{+}q)\,.
\end{aligned}$$

After a shift in l we confirm (A.9).

(B) Naive factorization means according to (A.1), using the shift invariance,

$$\begin{aligned}
&I^{(s)}_{\mu_1\ldots\mu_n}(p_0{-}p_s, p_1{-}p_s,\ldots,p_{s-1}{-}p_s; m_0,m_1,\ldots,m_{s-1}) \\
&\quad = g^{\nu_1\nu_2} I^{(s+1)}_{\mu_1\ldots\mu_n\nu_1\nu_2}(p_0{-}p_s,p_1{-}p_s,\ldots,p_{s-1}{-}p_s,0;m_0,m_1,\ldots,m_s) \\
&\quad\ - m_s^2 I^{(s+1)}_{\mu_1\ldots\mu_n}(p_0{-}p_s,p_1{-}p_s,\ldots,p_{s-1}{-}p_s,0;m_0,m_1,\ldots,m_s)\,.
\end{aligned}\tag{A.10}$$

For the leading terms in (A.7) and (A.8) this amounts to verify, when inserting the identity

$$g^{\mu_{n-1}\mu_n}T^k_{\mu_1\ldots\mu_n}(p) = (k{+}n{+}2)T^k_{\mu_1\ldots\mu_{n-2}}(p) + p^2 T^{k-2}_{\mu_1\ldots\mu_{n-2}}(p)\,,$$

the following equation

$$\begin{aligned}
&\int d^{s-1}x\sum_{\delta=0}^{[n/2]-s+2}\frac{(-1)^{n-\delta}}{2^{\delta+s-2}\delta!}T^{n-2\delta-2s+4}_{\mu_1\ldots\mu_n}\left((p_0{-}p_s)+\sum_{i=1}^{s-1}x_i(p_i{-}p_0)\right)\times \\
&\qquad\qquad \times\left\{\sum_{i,j=1}^{s-1}x_i(\delta_{ij}{-}x_j)(p_i{-}p_0)(p_j{-}p_0) - M^2_{s-1}(x)\right\}^{\delta} = \\
&= \int d^s x\sum_{\delta=0}^{[n/2]-s+2}\frac{(-1)^{n-\delta}}{2^{\delta+s-2}\delta!}(n{-}\delta{-}s{+}4)T^{n-2\delta-2s+4}_{\mu_1\ldots\mu_n}\left((p_0{-}p_s)+\sum_{i=1}^{s}x_i(p_i{-}p_0)\right)\times \\
&\qquad\qquad \times\left\{\sum_{i,j=1}^{s}x_i(\delta_{ij}{-}x_j)(p_i{-}p_0)(p_j{-}p_0) - M^2_s(x)\right\}^{\delta} + \qquad\qquad (A.11) \\
&+ \int d^s x\sum_{\delta=0}^{[n/2]-s+1}\frac{(-1)^{n-\delta}}{2^{\delta+s-1}\delta!}\left(\left((p_0{-}p_s)+\sum_{i=1}^{s}x_i(p_i{-}p_0)\right)^2 - m_s^2\right)T^{n-2\delta-2s+2}_{\mu_1\ldots\mu_n}\times
\end{aligned}$$

$$\times \left((p_0 - p_s) + \sum_{i=1}^{s} x_i (p_i - p_0)\right) \left\{\sum_{i,j=1}^{s} x_i (\delta_{ij} - x_j)(p_i - p_0)(p_j - p_0) - M_s^2(x)\right\}^{\delta} .$$

We have verified (A.11) for the values of $s \in \{1,2,3\}$ and $0 \leq n \leq 2s$ relevant for this paper by explicit calculation. Unfortunately we do not have a general proof. The integration over x_s on the r.h.s. of (A.11) leads to Appell hypergeometric functions which we did not succeed to treat.

References

[1] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[2] A. Konechny and A. Schwarz, *Introduction to M(atrix) theory and noncommutative geometry*, hep-th/0012145.

[3] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2002) 977 [hep-th/0106048].

[4] R.J. Szabo, *Quantum field theory on noncommutative spaces*, hep-th/0109162.

[5] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[6] T. Filk, *Divergencies in a field theory on quantum space*, *Phys. Lett.* **B 376** (1996) 53.

[7] A. Gonzalez-Arroyo and C.P. Korthals Altes, *Reduced model for large-N continuum field theories*, *Phys. Lett.* **B 131** (1983) 396.

[8] A. Connes, *Gravity coupled with matter and the foundation of non-commutative geometry*, *Commun. Math. Phys.* **182** (1996) 155 [hep-th/9603053].

[9] J.C. Várilly and J.M. Gracia-Bondía, *On the ultraviolet behaviour of quantum fields over noncommutative manifolds*, *Int. J. Mod. Phys.* **A 14** (1999) 1305 [hep-th/9804001].

[10] C.P. Martín and D. Sánchez-Ruiz, *The one-loop uv divergent structure of* U(1) *Yang-Mills theory on noncommutative R^4*, *Phys. Rev. Lett.* **83** (1999) 476 [hep-th/9903077].

[11] M.M. Sheikh-Jabbari, *Renormalizability of the supersymmetric Yang-Mills theories on the noncommutative torus*, *J. High Energy Phys.* **06** (1999) 015 [hep-th/9903107].

[12] T. Krajewski and R. Wulkenhaar, *Perturbative quantum gauge fields on the noncommutative torus*, *Int. J. Mod. Phys.* **A 15** (2000) 1011 [hep-th/9903187].

[13] L. Bonora and M. Salizzoni, *Renormalization of noncommutative* U(N) *gauge theories*, *Phys. Lett.* **B 504** (2001) 80 [hep-th/0011088].

[14] A. Armoni, *Comments on perturbative dynamics of non-commutative Yang-Mills theory*, *Nucl. Phys.* **B 593** (2001) 229 [hep-th/0005208].

[15] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, *J. High Energy Phys.* **02** (2000) 020 [hep-th/9912072].

[16] M. Hayakawa, *Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on R^4*, hep-th/9912167.

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[17] A. Matusis, L. Susskind and N. Toumbas, *The IR/UV connection in the non-commutative gauge theories*, *J. High Energy Phys.* **12** (2000) 002 [hep-th/0002075].

[18] I. Chepelev and R. Roiban, *Convergence theorem for non-commutative Feynman graphs and renormalization*, *J. High Energy Phys.* **03** (2001) 001 [hep-th/0008090].

[19] A.A. Bichl, J.M. Grimstrup, L. Popp, M. Schweda and R. Wulkenhaar, *Perturbative analysis of the Seiberg-Witten map*, hep-th/0102044.

[20] B. Jurčo, S. Schraml, P. Schupp and J. Wess, *Enveloping algebra valued gauge transformations for non-abelian gauge groups on non-commutative spaces*, *Eur. Phys. J.* **C 17** (2000) 521 [hep-th/0006246].

[21] A. Bichl et al., *Renormalization of the noncommutative photon selfenergy to all orders via Seiberg-Witten map*, *J. High Energy Phys.* **06** (2001) 013 [hep-th/0104097].

[22] A.A. Bichl, J.M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda and R. Wulkenhaar, *Noncommutative Lorentz symmetry and the origin of the Seiberg-Witten map*, hep-th/0108045.

[23] D. Colladay and V.A. Kostelecky, *Lorentz-violating extension of the standard model*, *Phys. Rev.* **D 58** (1998) 116002 [hep-ph/9809521].

[24] A.A. Bichl, J.M. Grimstrup, L. Popp, M. Schweda and R. Wulkenhaar, *Deformed QED via Seiberg-Witten map*, hep-th/0102103.

[25] E.R. Speer, *Dimensional and analytic renormalization*, in proceedings *Renormalization theory*, Erice 1975, Dordrecht 1976, p. 25–93.

[26] M. Ertl, *Index_0.14.2 — A Mathematica^{TM} package for index manipulations*, unpublished.

[27] J.M. Gracia-Bondía and J.C. Várilly, *Algebras of distributions suitable for phase space quantum mechanics, 1*, *J. Math. Phys.* **29** (1988) 869.

[28] J.M. Gracia-Bondía, F. Lizzi, G. Marmo and P. Vitale, *Infinitely many star products to play with*, hep-th/0112092.

[29] J.M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda and R. Wulkenhaar, *Noncommutative spin-1/2 representations*, hep-th/0110231.

[30] T. Asakawa and I. Kishimoto, *Comments on gauge equivalence in noncommutative geometry*, *J. High Energy Phys.* **11** (1999) 024 [hep-th/9909139].

[31] E. Langmann, *Generalized Yang-Mills actions from Dirac operator determinants*, *J. Math. Phys.* **42** (2001) 5238 [math-ph/0104011].

[32] W. Zimmermann, *Reduction in the number of coupling parameters*, *Commun. Math. Phys.* **97** (1985) 211.

[33] A. Connes and M. Dubois-Violette, *Noncommutative finite-dimensional manifolds, 1. Spherical manifolds and related examples*, math.QA/0107070.

[34] A.H. Chamseddine and A. Connes, *The spectral action principle*, *Commun. Math. Phys.* **186** (1997) 731 [hep-th/9606001].